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Quantum physics as the projective representation theory of Noether symmetries

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Abstract

I construct lowest-energy representations of non-centrally extended algebras of Noether symmetries, including diffeomorphisms and reparametrizations of the observer's trajectory. This may be viewed as a new scheme for quantization. First classical physics is formulated as the cohomology of a certain Koszul-Tate (KT) complex, using not only fields and antifields but also their conjugate momenta. Then all fields are expanded in a Taylor series around the observer's present position, and terms of order higher than p are truncated. Finally, quantization is carried out by replacing Poisson brackets by commutators and imposing the KT cohomology in Fock space. This procedure is consistent for finite p, but the limit $p \to \infty$ leads to difficulties.

1 Introduction

The main hypothesis underlying this work is that physics is the representation theory of its Noether symmetries, the most prominent ones being spacetime diffeomorphisms and reparametrizations of the observer's trajectory. After quantization one expects to find a projective representation of this group, i.e. a representation up to a local phase. On the Lie algebra level, this corresponds to an abelian but non-central extension of $diff(N) \oplus diff(1)$; only if the phase is globally constant, the Lie algebra extension is central. In [13], I discovered the "DRO (Diffeomorphism,

Reparametrization, Observer) algebra" $DRO(N)$ (the name, however, is new), which is a non-split abelian extension of $diff(N) \oplus diff(1)$ by the commutative algebra of local functionals on the observer's trajectory, depending on four parameters ("abelian charges"). This discovery builds on previous work by Eswara-Rao and Moody [6] and myself [11]. Related work goes under the name "toroidal Lie algebras" [1, 2, 4, 14]. The DRO algebra, and the more general "DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra" $DGRO(N, \mathfrak{g})$ obtained by adding an ordinary gauge algebra $map(N, \mathfrak{g})$ to the Noether symmetries, are described in section 2.

The main ingredient missing in [13] is that it makes no reference to dynamics (i.e. action, Hamiltonian, or Euler-Lagrange (EL) equations), so it is a purely kinematical theory. To introduce dynamics into the picture, I employ the following strategy. First the solutions to the classical equations of motion (EL and geodesic) are described in terms of the cohomology of a certain Koszul-Tate (KT) complex. This is closely related to the Batalin-Vilkovisky formalism, as formulated in [9, 16]. However, there is one important difference: I do not only introduce fields and antifields, but also field and antifield momenta. This has several implications: 1. The KT differential can be expressed as a Poisson bracket with a KT generator. 2. The antibracket is not a fundamental object. 3. The cohomology grows; it consists of differential forms (not just functions) on the stationary surface. Nevertheless, this enlarged KT cohomology still encodes classical dynamics, and it is the subject of section 3.

The next idea is to expand all fields and antifields, but not their momenta, in a Taylor series around the observer's present position, and to truncate after terms of order p, i.e. to pass to p-jet space. I now quantize in the naïve sense of the word: take a formulation of classical physics, replace all Poisson brackets by commutators and representent the resulting Heisenberg algebra on a unique Fock space. Although the fields do not depend on "parameter time" (i.e. the parameter along the observer's trajectory), the Taylor coefficients do. It is therefore possible to make a Fourier expansion of both the jets, the trajectory, and all momenta in parameter time, and proclaim that the Fock vacuum be annihilated by all negative energy modes. This step, and in particular the form of the resulting extensions, was the main result of [13]; it is reviewed in section 4.

The main advantage of the KT complex is that it survives quantization. In contradistinction to the BRST generator, the KT generator is bilinear in commuting variables, and thus already normal ordered. In section 5 I describe the KT generator in jet space and the associated quantum KT com-

plex. In particular, the action of the DGRO algebra on the cohomology is computed. The resulting $DGRO(N, \mathfrak{g})$ modules are manifestly well-defined quantum theories for all finite p. This can be viewed as an extreme statement of locality: the theory only deals with objects that are local to the observer, i.e. the fields and finitely many derivatives thereof at the observer's present position. This is not to say that events away from the observer are unphysical, but they are not described by the theory. To recover objective reality of distant events, we should demand that the limit $p \to \infty$ exists. The leading behaviour of the abelian charges is studied, but it is found to diverge due to second-order antifields. Some means to avoid this type of infinity are discussed, but none of these is satisfactory.

In the course of this work I introduce several modifications to the formalism of physics. These changes are dictated by the desire to obtain well-defined quantum representations of the Noether symmetries, but there remains to clarify the relation to standard formulations of quantum physics. However, even if my results turn out to be physically irrelevant, they are still of independent mathematical interest since new representations of naturally arising Lie algebras are constructed.

2 The algebras $DRO(N)$ and $DGRO(N, \mathfrak{g})$

Let $\xi = \xi^\mu(x)\partial_\mu$, $x \in \mathbb{R}^N$, $\partial_\mu = \partial/\partial x^\mu$, be a vector field, with commutator $[\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu$. Greek indices $\mu, \nu = 0, 1, .., N-1$ label the spacetime coordinates and the summation convention is used on all kinds of indices. The diffeomorphism algebra (algebra of vector fields, Witt algebra) $diff(N)$ is generated by Lie derivatives $\mathcal{L}_\xi$. In particular, we refer to diffeomorphisms on the circle as reparametrizations. They form an additional $diff(1)$ algebra with generators L_f, where $f = f(t)d/dt$, $t \in S^1$, is a vector field on the circle. The commutator is $[f, g] = (f\dot{g} - g\dot{f})d/dt$, where a dot indicates the t derivative. Moreover, introduce N priviledged functions on the circle $q^\mu(t)$, which can be interpreted as the trajectory of an observer (or base point). Let the observer algebra $Obs(N) = \mathbb{C}[q(t)]$ be the space of local functionals of $q^\mu(t)$, i.e. polynomial functions of $q^\mu(t)$, $\dot{q}^\mu(t)$, ... $d^k q^\mu(t)/dt^k$, k finite, regarded as a commutative Lie algebra.

The assumption that $t \in S^1$ is for technical simplicity; it enables jets to be expanded in a Fourier series, but it is physically quite unjustified because it means that spacetime is periodic in the time direction. However, all we really need is that $\int dt\ \dot{F}(t) = 0$ for all functions $F(t)$. Most results are

unchanged if we instead take $t \in \mathbb{R}$ and replace Fourier sums with Fourier integrals everywhere.

The *DRO (Diffeomorphism, Reparametrization, Observer)* algebra $DRO(N)$ is an abelian but non-central Lie algebra extension of $diff(N) \oplus diff(1)$ by $Obs(N)$:

$$0 \longrightarrow Obs(N) \longrightarrow DRO(N) \longrightarrow diff(N) \oplus diff(1) \longrightarrow 0. \tag{2.1}$$

The extension depends on the four parameters c_j, $j = 1, 2, 3, 4$, to be called *abelian charges*; the name is chosen in analogy with the central charge of the Virasoro algebra. The sequence (2.1) splits ($DRO(N)$ is a semi-direct product) iff all four abelian charges vanish. The brackets are given by

$$\begin{aligned}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi,\eta]} + \frac{1}{2\pi i}\int dt\ \dot q^\rho(t)\Big(c_1\partial_\rho\partial_\nu\xi^\mu(q(t))\partial_\mu\eta^\nu(q(t)) + \\
&\qquad + c_2\partial_\rho\partial_\mu\xi^\mu(q(t))\partial_\nu\eta^\nu(q(t))\Big), \\
[L_f, \mathcal{L}_\xi] &= \frac{c_3}{4\pi i}\int dt\ (\ddot f(t) - i\dot f(t))\partial_\mu\xi^\mu(q(t)), \\
[L_f, L_g] &= L_{[f,g]} + \frac{c_4}{24\pi i}\int dt(\ddot f(t)\dot g(t) - \dot f(t) g(t)), \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\mu(t)] &= -f(t)\dot q^\mu(t), \\
[q^\mu(s), q^\nu(t)] &= 0,
\end{aligned} \tag{2.2}$$

extended to all of $Obs(N)$ by Leibniz' rule and linearity. Two abelian charges have been renamed compared to [13]: $c_3 = c_0$ and $c_4 = c$, where c is the central charge in the Virasoro algebra generated by reparametrizations. Also, the value of a trivial cocycle has been fixed.

To prove that (2.2) defines a Lie algebra is straightforward; one either checks all Jacobi identities, or notes the existence of the explicit realization below. Non-triviality was not proven in [13], but this is easily rectified. The strategy is to consider the restriction of (2.2) to various subalgebras. The L_f generate a Virasoro algebra with central charge c_4, and the terms proportional to c_1 and c_2 are identified as extensions ψ_4^W and ψ_3^W in Dzhumadil'daev's classification [3]. To prove that c_3 is non-trivial, we set $c_1 = c_2 = c_4 = 0$ and consider the restriction to the subalgebra generated by $K_f = \mathcal{L}_\xi + L_f$, where $\xi = f(x^0)\partial_0$:

$$[K_f, K_g] = K_{[f,g]} + \frac{c_3}{4\pi i}\int dt\ (\ddot f(t) g'(q^0(t)) - f'(q^0(t))\ddot g(t)), \tag{2.3}$$

$$[K_f, q^0(t)] \;=\; f(q^0(t)) - \dot q^0(t) f(t),$$

apart from a trivial term. If we (consistently) set $q^0(t) = t$, (2.3) becomes a Virasoro algebra with central charge $12c_3$, and hence c_3 is non-trivial. Finally, we note that all four terms behave differently under the restrictions considered, so they must be inequivalent. Q.E.D.

It is not difficult to reformulate the DRO algebra as a proper Lie algebra, by introducing a compete basis for $Obs(N)$. In fact, it suffices to consider two infinite families of linear operators $S_n^{\nu_1..\nu_n}(F_{\nu_1..\nu_n})$, $R_n^{\rho|\nu_1..\nu_n}(G_{\rho|\nu_1..\nu_n})$, defined for arbitrary functions $F_{\nu_1..\nu_n}(t,x)$, $G_{\rho|\nu_1..\nu_n}(t,x)$, $t \in S^1$, $x \in \mathbb{R}^N$, totally symmetric in the indices $\nu_1..\nu_n$.

$$\begin{aligned}
S_n^{\nu_1..\nu_n}(F_{\nu_1..\nu_n}) &= \frac{1}{2\pi i}\int dt\ \dot q^{\nu_1}(t)..\dot q^{\nu_n}(t) F_{\nu_1..\nu_n}(t, q(t)), \\
R_n^{\rho|\nu_1..\nu_n}(G_{\rho|\nu_1..\nu_n}) &= \frac{1}{2\pi i}\int dt\ \ddot q^{\rho}(t)\dot q^{\nu_1}(t)..\dot q^{\nu_n}(t) G_{\rho|\nu_1..\nu_n}(t, q(t)).
\end{aligned} \tag{2.4}$$

Then $\mathcal{L}_\xi$, L_f, $S_n^{\nu_1..\nu_n}(F_{\nu_1..\nu_n})$, $R_n^{\rho|\nu_1..\nu_n}(G_{\rho|\nu_1..\nu_n})$ generate a Lie algebra, whose brackets are explicitly written down in [13].

Consider also the gauge (or current) algebra $map(N, \mathfrak{g})$, where $\mathfrak{g}$ is finite-dimensional Lie algebra with basis J^a (hermitian if $\mathfrak{g}$ is compact and semisimple), structure constants $f^{ab}{}_c$ and brackets $[J^a, J^b] = i f^{ab}{}_c J^c$. Our notation is similar to [8] or [7], chapter 13. We always assume that $\mathfrak{g}$ has a Killing metric proportional to δ^{ab}. Then there is no need to distinguish between upper and lower $\mathfrak{g}$ indices, and the structure constants $f^{abc} = \delta^{cd} f^{ab}{}_d$ are totally antisymmetric. Further assume that there is a priviledged vector $\delta^a \propto \mathrm{tr} J^a$, such that $f^{ab}{}_c \delta^c \equiv 0$. Of course, $\delta^a = 0$ if $\mathfrak{g}$ is semisimple, but it may be non-zero if $\mathfrak{g}$ contains abelian factors. The primary example is $\mathfrak{g} = gl(N)$, where $\mathrm{tr}(T^\mu_\nu) \propto \delta^\mu_\nu$.

Let $X = X_a(x) J^a$, $x \in \mathbb{R}^N$, be a $\mathfrak{g}$-valued function and define $[X, Y] = i f^{ab}{}_c X_a Y_b J^c$. $map(N, \mathfrak{g})$ is the algebra of maps from $\mathbb{R}^N$ to $\mathfrak{g}$. Its generators are denoted by $\mathcal{J}_X$. The *DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra* $DGRO(N, \mathfrak{g})$ has brackets

$$\begin{aligned}
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X,Y]} - \frac{c_5}{2\pi i}\delta^{ab}\int dt\ \dot q^\rho(t)\partial_\rho X_a(q(t)) Y_b(q(t)), \\
[L_f, \mathcal{J}_X] &= \frac{c_6}{4\pi i}\delta^a \int dt\ (\dddot f(t) - i\dot f(t)) X_a(q(t)), \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_{\xi^\mu\partial_\mu X} - \frac{c_7}{2\pi i}\delta^a\int dt\ \dot q^\rho(t) X_a(q(t))\partial_\rho\partial_\mu \xi^\mu(q(t)), \\
[\mathcal{J}_X, q^\mu(t)] &= 0,
\end{aligned} \tag{2.5}$$

in addition to (2.2). In [13], the constants were denoted by $k = c_5$ (the extended algebra reduces to the Kac-Moody algebra $\widehat{\mathfrak{g}}$ when $N = 1$ and $q^0(t) = t$), $g^a = c_6\delta^a$ and $g'^a = c_7\delta^a$. The present notation has the advantage that all abelian charges c_j, $j = 1, \ldots, 7$, can be discussed collectively.

It is sometimes better not to work with smeared generators, so we define $\mathcal{L}_\mu(x)$, $L(t)$ and $\mathcal{J}^a(x)$ by

$$\begin{aligned} \mathcal{L}_\xi &= \int d^N x\ \xi^\mu(x)\mathcal{L}_\mu(x), \\ L_f &= \int dt\ f(t)L(t), \\ \mathcal{J}_X &= \int d^N x\ X_a(x)\mathcal{J}^a(x). \end{aligned} \tag{2.6}$$

In [11] I described a gauge-fixed version of the DRO algebra, denoted by $\widetilde{diff}(N)$. To obtain it, we must recall Dirac's treatment of constrained Hamiltonian systems, and adapt it to Lie algebras [9]. Consider embeddings of some Lie algebra $\mathfrak{g}$ into the Poisson algebra $C^\infty(\mathcal{P})$, where $\mathcal{P}$ is a phase space. Let $P, R, ..$ label constraints χ_P, which are assumed bosonic for simplicity. Consider the constraint surface $\chi_P \approx 0$, where weak equality (i.e. equality modulo constraints) is denoted by $\approx$. Constraints are second class if the Poisson bracket matrix $C_{PR} = [\chi_P, \chi_R]$ is invertible; otherwise, they are first class and generate a Lie algebra. Assume that all constraints are second class. Then the matrix C_{PR} has an inverse, denoted by Δ^{PR}. The Dirac bracket

$$[A, B]^* = [A, B] - [A, \chi_P]\Delta^{PR}[\chi_R, B] \tag{2.7}$$

defines a new Poisson bracket which is compatible with the constraints: $[A, \chi_R]^* = 0$ for every $A \in \mathfrak{g}$. Of course, there is no guarantee that the operators A, B still generate the same Lie algebra under the Dirac brackets. A sufficient condition for this is that the constraints are preserved in the sense that $[A, \chi_P] \approx 0$ for every A. A less restrictive condition is often possible. Usually, the constraints can be divided into two sets $\chi_P = (\Phi_a, \Pi^a)$, such that $[\Phi^a, \Phi_b] \approx 0$. The Φ^a are then first class, and Π_a are gauge conditions. It is then sufficient that $[A, \Phi_a] \approx 0$, because the components of Δ^{PR} that involve Π's on both sides vanish.

Now consider the case $\mathfrak{g} = DRO(N)$. Strictly speaking, we can only pass to Dirac brackets if $\mathfrak{g}$ admits a Poisson bracket realization, which is not necessarily true in the presence of abelian extensions. If we ignore this

problem, reparametrizations and one component of the observer's trajectory can be eliminated by introduction of the second-class constraints

$$\chi(s) = \begin{pmatrix} q^0(s) \\ L(s) \end{pmatrix} \approx 0. \tag{2.8}$$

In the absense of extensions, $[\mathcal{L}_\xi, L(t)] = 0$, so this constraint is of the type above. When the extensions are turned on, new terms arise, but we still have an abelian extension of $diff(N)$. The Poisson bracket matrix $C(s,t)$ and its inverse $\Delta(s,t)$ are, on the constraint surface,

$$\begin{aligned}
C(s,t) &\equiv [\chi(s), \chi^T(t)] = \Big[\begin{pmatrix} q^0(s) \\ L(s) \end{pmatrix}, (\, q^0(t) \quad L(t)\,) \Big] \\
&\approx \begin{pmatrix} 0 & \delta(s-t) \\ -\delta(s-t) & \frac{c_4}{24\pi i}(\dddot{\delta}(s-t) + \dot{\delta}(s-t)) \end{pmatrix}, \\
\Delta(s,t) &\approx \begin{pmatrix} \frac{c_4}{24\pi i}(\dddot{\delta}(s-t) + \dot{\delta}(s-t)) & -\delta(s-t) \\ \delta(s-t) & 0 \end{pmatrix}.
\end{aligned} \tag{2.9}$$

We find

$$\begin{aligned}
[\mathcal{L}_\xi, \mathcal{L}_\eta]^* &= \mathcal{L}_{[\xi,\eta]} + \frac{1}{2\pi i} \int dt\ c_1 \partial_\nu \dot{\xi}^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + \\
&\quad + c_2 \partial_\mu \dot{\xi}^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) + \\
&\quad + \frac{c_3}{4\pi i} \int dt\ \partial_\nu \eta^\nu(q(t))(\ddot{\xi}^0(q(t)) - i\dot{\xi}^0(q(t))) - \\
&\quad - \partial_\mu \xi^\mu(q(t))(\ddot{\eta}^0(q(t)) - i\dot{\eta}^0(q(t))) + \\
&\quad + \frac{c_4}{24\pi i} \int dt\ \dddot{\xi}^0(q(t))\dot{\eta}^0(q(t)) - \dot{\xi}^0(q(t))\eta^0(q(t)), \\
[\mathcal{L}_\xi, q^\mu(t)]^* &= \xi^\mu(q(t)) - \dot{q}^\mu(t)\xi^0(q(t)), \\
[q^\mu(s), q^\nu(t)]^* &= 0, \\
[L(s), \mathcal{L}_\xi]^* &= [L(s), L(t)]^* = [L(s), q^\mu(t)]^* = 0.
\end{aligned} \tag{2.10}$$

Note that $[\mathcal{L}_\xi, q^0(t)]^* = 0$. This algebra deserves to be called the gauge-fixed diffeomorphism algebra, denoted by $\widetilde{diff}(N)$. Again, (2.10) is not a Lie algebra, but can be made so by introducing the new generators (2.4) [11].

3 Classical physics as Koszul-Tate cohomology

3.1 Configuration space and phase space

The configuration space $\mathcal{Q}$ is the space spanned by the observer's trajectory $q^\mu(t)$, $t \in S^1$, the einbein $e(t)$, $t \in S^1$, and a collection of V-valued fields over spacetime, where V carries a finite-dimensional $gl(N)$ representation ϱ. The fields are collectively denoted by $\phi_\alpha(x)$, $x \in \mathbb{R}^N$, where the V index α labels different tensor and internal components. If V contains several (bosonic or fermionic) field species, α labels these as well; in this case $\varrho = \varrho_1 \oplus \ldots \oplus \varrho_n$ is a direct sum.

In our convention, $gl(N)$ has basis T^μ_ν and brackets

$$[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\nu T^\mu_\tau - \delta^\mu_\tau T^\sigma_\mu. \tag{3.1}$$

To these elements correspond the matrices $\varrho(T^\nu_\mu)$ with elements $\varrho^\alpha_\beta(T^\nu_\mu)$. In particular, denote by $\varrho = (p, q; \kappa)$ the representation on tensor densities with p upper and q lower indices and weight κ:

$$\varrho(T^\mu_\nu)\phi^{\sigma_1..\sigma_p}_{\tau_1..\tau_q} = -\kappa\delta^\mu_\nu\phi^{\sigma_1..\sigma_p}_{\tau_1..\tau_q} + \sum_{i=1}^{p}\delta^{\sigma_i}_\nu\phi^{\sigma_1..\mu..\sigma_p}_{\tau_1..\tau_q} - \sum_{j=1}^{q}\delta^\mu_{\tau_j}\phi^{\sigma_1..\sigma_p}_{\tau_1..\nu..\tau_q}. \tag{3.2}$$

From $\mathcal{Q}$ we construct the corresponding phase space $\mathcal{P}$ by adjoining the conjugate momenta $p_\mu(t)$, $\pi_e(t)$ and $\pi^\alpha(x)$. The only non-zero Poisson brackets are

$$\begin{aligned}
[p_\nu(s), q^\mu(t)] &= \delta^\mu_\nu\delta(s-t), \\
[\pi_e(s), e(t)] &= \delta(s-t), \\
[\pi^\alpha(x), \phi_\beta(y)] &= -(-)^{\alpha\beta}[\phi_\beta(y), \pi^\alpha(x)] = \delta^\alpha_\beta\delta^N(x-y),
\end{aligned} \tag{3.3}$$

where $(-)^\alpha = 1$ ($(-)^\alpha = -1$) if ϕ_α is bosonic (fermionic) and $(-)^{\alpha\beta} = (-)^\alpha(-)^\beta$; the trajectory and einbein are both bosonic. Denote by $C^\infty(\mathcal{Q})$ and $C^\infty(\mathcal{P})$ the spaces of local functionals over $\mathcal{Q}$ and $\mathcal{P}$; (anti)symmetrization is automatically taken into account by the bosonic (fermionic) character of the fields. Then

$$\begin{aligned}
\mathcal{L}_\xi &= \int dt\ \xi^\mu(q(t))p_\mu(t) - \\
&\quad - \int d^Nx\ (\xi^\mu(x)\partial_\mu\phi_\alpha(x) + \partial_\nu\xi^\mu(x)\varrho^\beta_\alpha(T^\nu_\mu)\phi_\beta(x))\pi^\alpha(x) \qquad (3.4) \\
L_f &= \int dt\ f(t)(-\dot{q}^\mu(t)p_\mu(t) + e(t)\dot{\pi}_e(t)),
\end{aligned}$$

defines an embedding $DRO(N) \hookrightarrow C^\infty(\mathcal{P})$.

Consequently, (3.4) defines a $DRO(N)$ realization (by graded Poisson brackets) on $C^\infty(\mathcal{Q})$ and $C^\infty(\mathcal{P})$. Explicitly,

$$\begin{aligned}
[\mathcal{L}_\xi, \phi_\alpha(x)] &= -\xi^\mu(x)\partial_\mu\phi_\alpha(x) - \partial_\nu\xi^\mu(x)\varrho^\beta_\alpha(T^\nu_\mu)\phi_\beta(x), \\
[L_f, \phi_\alpha(x)] &= 0, \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\mu(t)] &= -f(t)\dot{q}^\mu(t), \\
[\mathcal{L}_\xi, e(t)] &= 0, \\
[L_f, e(t)] &= -f(t)\dot{e}(t) - \dot{f}(t)e(t),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
[\mathcal{L}_\xi, \pi^\alpha(x)] &= -\xi^\mu(x)\partial_\mu\pi^\alpha(x) + \partial_\nu\xi^\mu(x)\pi^\beta(x)\varrho^\alpha_\beta(T^\nu_\mu), \\
[L_f, \pi^\alpha(x)] &= 0, \\
[\mathcal{L}_\xi, p_\nu(t)] &= -\partial_\nu\xi^\mu(q(t))p_\mu(t), \\
[L_f, p_\nu(t)] &= -f(t)\dot{p}_\nu(t) - \dot{f}(t)p_\nu(t), \\
[\mathcal{L}_\xi, \pi_e(t)] &= 0, \\
[L_f, \pi_e(t)] &= -f(t)\dot{\pi}_e(t).
\end{aligned} \tag{3.6}$$

3.2 Euler-Lagrange constraint

Let

$$S[\phi] = \int d^Nx \ \sqrt{|g(x)|}\,\mathcal{L}(x;\phi) \tag{3.7}$$

be an action invariant under $diff(N)$, where the Lagrangian $\mathcal{L}(x;\phi)$ is a $diff(N)$ scalar field of weight zero (not to be confused with the Lie derivative $\mathcal{L}_\xi$). The notation emphasizes that the Lagrangian is a local functional of ϕ, i.e. a function of $\phi_\alpha(x)$ and finitely many derivatives at the spacetime point x. Moreover, $|g(x)| = \epsilon^{\mu_1\mu_2..\mu_N}\epsilon^{\nu_1\nu_2..\nu_N}g_{\mu_1\nu_1}(x)g_{\mu_2\nu_2}(x)\dots g_{\mu_N\nu_N}(x)$ is the determinant of the metric, although the only important point is that $\sqrt{|g(x)|}$ has weight one.

The solutions to the Euler-Lagrange (EL) equations,

$$\mathcal{E}^\alpha(x;\phi) \equiv [\pi^\alpha(x), S] \equiv \frac{\delta S}{\delta\phi_\alpha(x)} = 0, \tag{3.8}$$

define the *stationary surface* $\Sigma \subset \mathcal{Q}$. The EL equations also generate the multiplicative ideal $\mathcal{N}_S = \{f_\alpha\mathcal{E}^\alpha(x;\phi) : f_\alpha \in C^\infty(\mathcal{Q})\} \subset C^\infty(\mathcal{Q})$. The factor

space $C^\infty(\mathcal{Q})/\mathcal{N}_S$ can be identified with the algebra of local functionals on the stationary surface, i.e. $C^\infty(\Sigma)$. Conversely, Σ can be recovered as the set of maximal ideals of $C^\infty(\Sigma)$, so knowledge of this algebra is equivalent to solving the EL equations. The problem is now to describe $C^\infty(\Sigma) = C^\infty(\mathcal{Q})/\mathcal{N}_S$ in a simple manner. This space admits a resolution in terms of a certain Koszul-Tate (KT) complex [9]; the present exposition was mainly inspired by Stasheff [16]. Recall that a complex $\Omega^\bullet$ is a collection of spaces Ω^g and maps δ_g,

$$\ldots \xrightarrow{\delta_{-3}} \Omega^{-2} \xrightarrow{\delta_{-2}} \Omega^{-1} \xrightarrow{\delta_{-1}} \Omega^0 \xrightarrow{\delta_0} \Omega^1 \xrightarrow{\delta_1} \Omega^2 \ldots, \tag{3.9}$$

such that $\delta_g \delta_{g-1} = 0$. The cohomology spaces are $H^g(\delta) = \ker \delta_g / \mathrm{im}\, \delta_{g-1}$. The complex (3.9) yields a one-sided resolution of a space V if $H^\ell(\delta) = V$, $\Omega^g = 0$ if $g > \ell$, and $H^g(\delta) = 0$ if $g < \ell$.

For each component of the EL equation $\mathcal{E}^\alpha(x)$ (the functional dependence on ϕ is henceforth suppressed), introduce an *antifield* $\phi^{*\alpha}(x)$ with Grassmann parity opposite to $\phi_\alpha(x)$ (and $\mathcal{E}^\alpha(x)$). Assign ghost numbers $\mathrm{gh}\, \phi_\alpha(x) = 0$, $\mathrm{gh}\, \phi^{*\alpha}(x) = -1$. The term "ghost number" is perhaps somewhat misleading, since I never introduce any ghosts, but the name is chosen in analogy with the Batalin-Vilkovisky terminology, see subsection 3.9 below. The KT complex is the space $\Omega^\bullet_{KT} = C^\infty(\mathcal{Q}) \otimes \mathbb{C}[b]$, where the second factor consists of local polynomial functionals in the antifields; (anti-)symmetrization is automatically taken care of by the (anti-)commuting nature of the antifields. This complex is naturally graded by ghost number, and there is a nilpotent KT differential δ, defined by

$$\delta\phi_\alpha(x) = 0, \qquad \delta\phi^{*\alpha}(x) = \mathcal{E}^\alpha(x). \tag{3.10}$$

Since $\ker \delta_0 = C^\infty(\mathcal{Q})$ and $\mathrm{im}\, \delta_{-1} = \mathcal{N}_S$, $H^0(\delta) = C^\infty(\mathcal{Q})/\mathcal{N}_S = C^\infty(\Sigma)$ as desired. Moreover, in the absense of Noether identities, $H^g(\delta) = 0$, $g < 0$, and $\Omega^g = 0$, $g > 0$, so we have obtained a resolution of $C^\infty(\Sigma)$.

For each antifield $\phi^{*\alpha}(x)$, introduce an antifield momentum $\pi^*_\alpha(x)$ satisfying Poisson brackets

$$[\pi^*_\alpha(x), \phi^{*\beta}(y)] = \delta^\beta_\alpha \delta^N(x-y). \tag{3.11}$$

Since the antifield has opposite Grassman parity compared to $\phi_\alpha(x)$, this bracket is symmetric if the original field is bosonic and vice versa. The KT differential can now be expressed as

$$\delta f = [Q_{KT}, f], \qquad \forall f \in C^\infty(\mathcal{Q}) \otimes \mathbb{C}[b], \tag{3.12}$$

where the fermionic KT generator Q_{KT} is

$$Q_{KT} = \int d^Nx\ \mathcal{E}^\alpha(x)\pi^*_\alpha(x). \tag{3.13}$$

$[Q_{KT}, Q_{KT}] = 0$ because $\mathcal{E}^\alpha(x)$ commutes with the antifield momentum. Formula (3.12) extends δ to the phase space analogue of the KT complex, $\Omega^\bullet_{KT} = C^\infty(\mathcal{P}) \otimes \mathbb{C}[\phi^*, \pi^*]$ (polynomial functionals in antifields and antifield momenta):

$$\delta\pi^\alpha(x) = -\int d^Ny\ (-)^\alpha \frac{\delta\mathcal{E}^\beta(y)}{\delta\phi_\alpha(x)}\pi^*_\beta(y), \qquad \delta\pi^*_\alpha(x) = 0, \tag{3.14}$$

where $\delta\mathcal{E}^\beta(y)/\delta\phi_\alpha(x) = [\pi^\alpha(x), \mathcal{E}^\beta(y)]$.

Before evaluating the cohomology, we note that $\Omega^\bullet_{KT}$ admits a double grading. Assign ghost numbers by

$$\begin{aligned} &\mathrm{gh}\,\phi^{*\alpha}(x) = -1, \qquad \mathrm{gh}\,\pi^*_\alpha(x) = +1, \\ &\mathrm{gh}\,\phi_\alpha(x) = \mathrm{gh}\,\pi^\alpha(x) = \mathrm{gh}\,q^\mu(t) = \mathrm{gh}\,p_\mu(t) = \mathrm{gh}\,e(t) = \mathrm{gh}\,\pi_e(t) = 0. \end{aligned} \tag{3.15}$$

We have $\mathrm{gh}\,[f,g] = \mathrm{gh}\,f + \mathrm{gh}\,g$ and $\mathrm{gh}\,Q = +1$. We may write $\mathrm{gh}\,f = [N_{\mathrm{gh}}, f]$, where the ghost number operator $N_{\mathrm{gh}} = -\int d^Nx\ \phi^{*\alpha}(x)\pi^*_\alpha(x)$. Moreover, introduce the *momentum number* by

$$\begin{aligned} &\mathrm{mom}\,\pi^\alpha(x) = \mathrm{mom}\,\pi^*_\alpha(x) = \mathrm{mom}\,p_\mu(t) = \mathrm{mom}\,\pi_e(t) = 1, \\ &\mathrm{mom}\,\phi_\alpha(x) = \mathrm{mom}\,\phi^{*\alpha}(x) = \mathrm{mom}\,q^\mu(t) = \mathrm{mom}\,e(t) = 0. \end{aligned} \tag{3.16}$$

We have $\mathrm{mom}\,[f,g] = \mathrm{mom}\,f + \mathrm{mom}\,g - 1$ and $\mathrm{mom}\,Q = +1$, but contrary to gh, mom can not be expressed in bracket form.

$DRO(N)$ acts as follows on $\Omega^\bullet_{KT}$: $\mathcal{E}^\alpha(x)$ transforms as $\pi^\alpha(x)$, the antifields are defined to transform in the same way, and thus the antifield momenta behave like $\phi_\alpha(x)$.

$$\begin{aligned} [\mathcal{L}_\xi, \mathcal{E}^\alpha(x)] &= -\xi^\mu(x)\partial_\mu\mathcal{E}^\alpha(x) + \partial_\nu\xi^\mu(x)\mathcal{E}^\beta(x)\varrho^\alpha_\beta(T^\nu_\mu), \\ [\mathcal{L}_\xi, \phi^{*\alpha}(x)] &= -\xi^\mu(x)\partial_\mu\phi^{*\alpha}(x) + \partial_\nu\xi^\mu(x)\phi^{*\beta}(x)\varrho^\alpha_\beta(T^\nu_\mu), \\ [\mathcal{L}_\xi, \pi^*_\alpha(x)] &= -\xi^\mu(x)\partial_\mu\pi^*_\alpha(x) - \partial_\nu\xi^\mu(x)\varrho^\beta_\alpha(T^\nu_\mu)\pi^*_\beta(x), \\ [L_f, \mathcal{E}^\alpha(x)] &= [L_f, \phi^{*\alpha}(x)] = [L_f, \pi^*_\alpha(x)] = 0. \end{aligned} \tag{3.17}$$

Hence the antifield contribution to the $DRO(N)$ generators is

$$\begin{aligned} \mathcal{L}^{(1)}_\xi &= \int d^Nx\ (-\xi^\mu(x)\partial_\mu\phi^{*\alpha}(x) + \partial_\nu\xi^\mu(x)\phi^{*\beta}(x)\varrho^\alpha_\beta(T^\nu_\mu))\pi^*_\alpha(x), \\ L^{(1)}_f &= 0. \end{aligned} \tag{3.18}$$

$\mathrm{gh}\,\mathcal{L}_\xi = \mathrm{gh}\,L_f = 0$ and $\mathrm{mom}\,\mathcal{L}_\xi = \mathrm{mom}\,L_f = 1$, which means that $DRO(N)$ commutes with the KT generator Q_{KT} and the momentum number is preserved.

The KT complex has the double decomposition

$$\Omega^\bullet_{KT} = \sum_{g=-\infty}^{\infty} \sum_{\ell=-\infty}^{g} \Omega^g_\ell, \tag{3.19}$$

where $\mathrm{gh}\,\Omega^g_\ell = g$, $\mathrm{mom}\,\Omega^g_\ell = \ell$. Since the ghost and momentum numbers are preserved, each cohomology group $H^g_\ell(Q_{KT})$ is separately a $DRO(N)$ module. The case $\ell = 0$ was described above: $H^g_0(Q_{KT}) = \delta^g_0 C^\infty(\mathcal{Q})/\mathcal{N}_S$. Similarly, $H^g_\ell(Q_{KT}) = 0$ if $g < 0$, and $H^\ell_\ell(Q_{KT})$ is the space of local functionals of $\phi(x)$ and $\pi^*_\alpha(x)$ of the form

$$f^{\alpha_1..\alpha_\ell}(\phi)\pi^*_{\alpha_1}\dots\pi^*_{\alpha_\ell}, \tag{3.20}$$

modulo the ideal generated by relations

$$\mathcal{E}^\alpha(x) = 0 \qquad \text{and} \qquad -\int d^N y\ (-)^\alpha \frac{\delta\mathcal{E}^\beta(y)}{\delta\phi_\alpha(x)}\pi^*_\beta(y) = 0. \tag{3.21}$$

The expression (3.20) is recognized as an ℓ-form over $\mathcal{Q}$. Since the antifield momenta commute with the fields and anticommute among themselves (for bosonic degrees of freedom), they can be thought of as differentials; schematically, $\pi^*_\alpha = d\phi_\alpha$. We have thus obtained a resolution of the space of ℓ-forms on the stationary surface Σ, and hence another description of Σ itself.

3.3 Auxiliary fields

There is considerable freedom to describe the cohomology spaces in non-minimal ways, by introducing auxiliary fields that are completely specified by their EL equations. An action the form $S^{(1)}[\phi,\psi]$, where $\psi_A(x) \equiv f_A(x;\phi)$, gives rise to the same KT cohomology as

$$S^{(2)}[\phi,\psi,\lambda] = S^{(1)}[\phi,\psi] + \int d^N x\ \lambda^A(x)(\psi_A(x) - f_A(x;\phi)), \tag{3.22}$$

where ψ_A is treated as an independent field and λ^A is a Lagrangian multiplier field. The EL equations for λ^A and ψ_A,

$$\psi_A(x) = f_A(x;\phi), \qquad \lambda^A(x) = -\frac{\delta S^{(1)}}{\delta\psi_A(x)}, \tag{3.23}$$

leave the same EL equations for ϕ_α:

$$\frac{\delta S^{(2)}}{\delta\phi_\alpha(x)} = \frac{\delta S^{(1)}}{\delta\phi_\alpha(x)} + \int d^N y\ \frac{\delta S^{(1)}}{\delta\psi_A(y)}\frac{\delta f_A(y)}{\delta\phi_\alpha(x)} = 0, \tag{3.24}$$

where (3.23) was used in the first step. Therefore, the cohomologies defined by the KT generators

$$\begin{aligned} Q^{(1)}_{KT} &= \int d^N x\ (\frac{\delta S^{(1)}}{\delta\phi_\alpha(x)} + \int d^N y\ \frac{\delta S^{(1)}}{\delta\psi_A(y)}\frac{\delta f_A(y)}{\delta\phi_\alpha(x)})\pi^*_\alpha(x), \\ Q^{(2)}_{KT} &= \int d^N x\ \Big((\frac{\delta S^{(1)}}{\delta\phi_\alpha(x)} - \int d^N y\ \lambda^A(y)\frac{\delta f_A(y)}{\delta\phi_\alpha(x)})\pi^*_\alpha(x) + \\ &\quad +(\lambda^A(x) - \frac{\delta S^{(1)}}{\delta\psi_A(x)})\frac{\delta}{\delta\psi^{*A}(x)} + (\psi_A(x) - f_A(x))\frac{\delta}{\delta\lambda^*_A(x)}\Big), \end{aligned} \tag{3.25}$$

are identical. Here $\delta/\delta\psi^{*A}$ and $\delta/\delta\lambda^*_A$ are the antifield momenta corresponding to ψ_A and λ^A, respectively.

In the main cases of physical interest, the KT generator can be made polynomial in all fields, provided that sufficiently many auxiliary fields are included. The following examples define some auxiliary fields that are needed below. Henceforth, they are tacitly assumed to be eliminated in cohomology by their EL equations.

1. The metric field $g_{\mu\nu}(x)$ has an inverse $g^{\mu\nu}(x)$, which can be regarded as an auxiliary field obeying the equation

$$g_{\mu\rho}(x)g^{\rho\nu}(x) = \delta^\nu_\mu. \tag{3.26}$$

2. The weight one field $\upsilon(x) = \sqrt{|g(x)|}$ used to densitize the Lagrangian can be eliminated by $\upsilon(x)^2 = |g(x)|$.

3. The Levi-Cività connection is given by the usual formula

$$\Gamma^\nu_{\sigma\tau}(x) = \frac{1}{2}g^{\nu\rho}(x)(\partial_\sigma g_{\rho\tau}(x) + \partial_\tau g_{\sigma\rho}(x) - \partial_\rho g_{\sigma\tau}(x)). \tag{3.27}$$

It verifies

$$\begin{aligned} [\mathcal{L}_\xi, \Gamma^\nu_{\sigma\tau}(x)] &= -\xi^\mu(x)\partial_\mu\Gamma^\nu_{\sigma\tau}(x) + \partial_\rho\xi^\nu(x)\Gamma^\rho_{\sigma\tau}(x) \\ &\quad -\partial_\sigma\xi^\mu(x)\Gamma^\nu_{\mu\tau}(x) - \partial_\tau\xi^\mu(x)\Gamma^\nu_{\sigma\mu}(x) - \partial_\sigma\partial_\tau\xi^\nu(x), \\ [L_f, \Gamma^\nu_{\sigma\tau}(x)] &= 0. \end{aligned} \tag{3.28}$$

We can now define the covariant (w.r.t. diffeomorphisms) derivative

$$\nabla_\mu = \partial_\mu - \Gamma^\rho_{\nu\mu}(x)\varrho(T^\nu_\rho). \tag{3.29}$$

4. The inverse of the einbein $e^{-1}(t)$, defined by $e^{-1}(t)e(t) = 1$.
5. The reparametrization connection $\Gamma(t) = -e^{-1}(t)\dot{e}(t)$, transforming as

$$\begin{aligned} [L_f, \Gamma(t)] &= -\dot{f}(t)\Gamma(t) - f(t)\dot{\Gamma}(t) + \ddot{f}(t), \\ [\mathcal{L}_\xi, \Gamma(t)] &= 0. \end{aligned} \tag{3.30}$$

Just as the Levi-Cività connection can be used to define a derivative which is covariant w.r.t. $diff(N)$, $\Gamma(t)$ is needed to define a derivative covariant w.r.t. reparametrizations $diff(1)$.

3.4 Geodesic constraint

Just as the fields are restricted to Cauchy data by the EL equations, the observer's trajectory can be eliminated by the geodesic equation, and the einbein is an auxiliary field satisfying

$$e(t) = \sqrt{g_{\mu\nu}(q(t))\dot{q}^\mu(t)\dot{q}^\nu(t)}. \tag{3.31}$$

These equations can also be cast in EL form. Add to (3.7) the term

$$S^{(q)}[q, e, g] = -\frac{1}{2}\int dt\ e(t) + e^{-1}(t)g_{\mu\nu}(q(t))\dot{q}^\mu(t)\dot{q}^\nu(t), \tag{3.32}$$

so the total action is $S[\phi, q, e] = S[\phi] + S^{(q)}[q, e, g]$. Note that $S^{(q)}$ depends on the metric, which is included in the set of fields. Define

$$\begin{aligned} \mathcal{G}_\mu(t) &\equiv [p_\mu(t), S] \\ &= e^{-1}(t)g_{\mu\nu}(q(t))(\ddot{q}^\nu(t) + \Gamma(t)\dot{q}^\nu(t) + \Gamma^\nu_{\sigma\tau}(q(t))\dot{q}^\sigma(t)\dot{q}^\tau(t)), \\ \mathcal{O}(t) &\equiv [\pi_e(t), S] \\ &= \frac{1}{2}(e^{-2}(t)g_{\mu\nu}(q(t))\dot{q}^\mu(t)\dot{q}^\nu(t) - 1), \end{aligned} \tag{3.33}$$

where $\Gamma(t)$ is the reparametrization connection (3.30). These operators transform homogeneously:

$$[\mathcal{L}_\xi, \mathcal{G}_\nu(t)] = -\partial_\nu\xi^\mu(q(t))\mathcal{G}_\mu(t), \tag{3.34}$$

$$
\begin{aligned}
[L_f, \mathcal{G}_\nu(t)] &= -f(t)\dot{\mathcal{G}}_\nu(t) - \dot{f}(t)\mathcal{G}_\nu(t), \\
\\
[\mathcal{L}_\xi, \mathcal{O}(t)] &= 0, \\
[L_f, \mathcal{O}(t)] &= -f(t)\dot{\mathcal{O}}(t).
\end{aligned}
\tag{3.35}
$$

We now introduce the trajectory antifield $q^*_\mu(t)$, with momentum $p^{*\mu}(t)$, and the einbein antifield $e^*(t)$, with momentum $\pi^*_e(t)$. Since $\mathcal{G}_\nu(t)$ and $\mathcal{O}(t)$ are bosonic, these antifields are fermionic and obey the non-zero anticommutation relations

$$
[p^{*\mu}(s), q^*_\nu(t)] = \delta^\mu_\nu \delta(s-t), \qquad [\pi^*_e(s), e^*(t)] = \delta(s-t). \tag{3.36}
$$

Ghost and momentum numbers are given by by

$$
\begin{aligned}
\mathrm{gh}\, q^*_\nu(t) = \mathrm{gh}\, e^*(t) = -1, \qquad & \mathrm{gh}\, p^{*\mu}(t) = \mathrm{gh}\, \pi^*_e(t) = +1, \\
\mathrm{mom}\, q^*_\nu(t) = \mathrm{mom}\, e^*(t) = 0, \qquad & \mathrm{mom}\, p^{*\mu}(t) = \mathrm{mom}\, \pi^*_e(t) = 1,
\end{aligned}
\tag{3.37}
$$

By adding the term

$$
Q^{(\mathcal{G})}_{KT} = \int dt\ \mathcal{G}_\mu(t) p^{*\mu}(t) + \mathcal{O}(t)\pi^*_e(t) \tag{3.38}
$$

to the KT differential, the constraints $\mathcal{G}_\nu(t) \approx \mathcal{O}(t) \approx 0$ are implemented in cohomology.

3.5 Noether identities

The previous discussion ignored the possibility of relations between the EL equations. This is certainly incorrect; at the very least, the DRO algebra imply certain conditions. In general we assume that there are Noether identities of the form

$$
r^a(x) \equiv \int d^N y\ r^a_\alpha(x,y)\mathcal{E}^\alpha(y) = \int d^N y\ (-)^\alpha \mathcal{E}^\alpha(y) r^a_\alpha(x,y) \equiv 0. \tag{3.39}
$$

For simplicity, let all Noether identities be independent; the addition of non-trivial relations between them is straightforward but leads to unnecessary complications. For each Noether identity, introduce a Noether (or second-order) antifield $\mathfrak{b}^a(x)$ with momentum $\mathfrak{c}_a(x)$. We only deal with bosonic Noether identities, and require their antifields to be bosonic as well. The non-zero Poisson bracket is

$$
[\mathfrak{c}_a(x), \mathfrak{b}^b(y)] = \delta^b_a \delta^N(x-y). \tag{3.40}
$$

A new (fermionic) term has to be added to the KT generator (3.13)

$$Q^{(2)}_{KT} = \iint d^N x \ d^N y \ (-)^\alpha r^a_\alpha(x,y)\phi^{*\alpha}(y)\mathfrak{c}_a(x). \tag{3.41}$$

The modified KT differential acts as

$$\begin{aligned}
[Q_{KT}, \phi_\alpha(x)] &= 0, \\
[Q_{KT}, \phi^{*\alpha}(x)] &= \mathcal{E}^\alpha(x), \\
[Q_{KT}, \mathfrak{b}^a(x)] &= \int d^N y \ (-)^\alpha r^a_\alpha(x,y)\phi^{*\alpha}(y),
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
[Q_{KT}, \pi^\alpha(x)] &= -\int d^N y \ (-)^\alpha \frac{\delta\mathcal{E}^\beta(y)}{\delta\phi_\alpha(x)}\pi^*_\beta(y) - \\
&\quad - \int d^N y \ d^N z \ (-)^{\alpha+\beta}\frac{\delta r^a_\beta(y,z)}{\delta\phi_\alpha(x)}\phi^{*\beta}(z)\mathfrak{c}_a(y), \\
[Q_{KT}, \pi^*_\alpha(x)] &= \int d^N y \ r^a_\alpha(y,x)\mathfrak{c}_a(y), \\
[Q_{KT}, \mathfrak{c}_a(x)] &= 0,
\end{aligned} \tag{3.43}$$

It follows from (3.39) that the KT generator is still nilpotent:

$$[Q_{KT}, Q_{KT}] = 2\iint d^N x \ d^N y \ (-)^\alpha \mathcal{E}^\alpha(x) r^a_\alpha(x,y)\mathfrak{c}_a(y) \equiv 0. \tag{3.44}$$

The addition of Noether antifields is necessary because we want the KT complex to yield a resolution, i.e. $H^g_\ell(Q_{KT}) = 0$ if $g \neq \ell$. In the presence of Noether identities,

$$\delta \int d^N y \ (-)^\alpha r^a_\alpha(x,y)\phi^{*\alpha}(y) = \int d^N y \ r^a_\alpha(x,y)\mathcal{E}^\alpha(y) \equiv 0, \tag{3.45}$$

so $\ker \delta_{-1} \neq 0$, but this expression is exact by (3.42), so $H^{-1}(Q_{KT})$ still vanishes.

3.6 Gauge symmetries

As is well known, Noether identities are connected to gauge symmetries. From (3.8) and (3.39) immediately follows that

$$\mathcal{J}_X = \iint d^N x \ d^N y \ X_a(x) r^a_\alpha(x,y)\pi^\alpha(y) \tag{3.46}$$

satisfies $[\mathcal{J}_X, S] = 0$. The set of such operators generate a Lie algebra, which is easily seen as follows. If $[\mathcal{J}_X, S] = [\mathcal{J}_Y, S] = 0$, $[[\mathcal{J}_X, \mathcal{J}_Y], S] = 0$ by the Jacobi identities. If some Noether identity were fermionic, (3.46) would define a super-Lie algebra, but this possibility is not considered here. Note that we use the same notation as for the proper gauge algebra $map(N, \mathfrak{g})$, but the present exposition is more general; in particular, it includes $DRO(N)$. This overloading should not cause confusion.

Assume that the Noether algebra can be written in localized form as

$$[\mathcal{J}^a(x), \mathcal{J}^b(y)] = \int d^N z\ f^{ab}{}_c(x, y; z)\mathcal{J}^c(z), \tag{3.47}$$

where

$$\mathcal{J}^a(x) = \int d^N y\ r^a_\alpha(x, y)\pi^\alpha(y) \tag{3.48}$$

and the structure constants $f^{ab}{}_c(x, y; z)$ depends on (finite derivatives of) $\delta(x - z)$ and $\delta(y - z)$ only. This is an assumption about locality which is always valid. Then the following identity holds

$$\begin{aligned}&\int d^N z\ r^a_\alpha(x, z)\frac{\delta r^b_\beta(y, w)}{\phi_\alpha(z)} - r^b_\alpha(y, z)\frac{\delta r^a_\beta(x, w)}{\phi_\alpha(z)} \\ &= \int d^N z\ f^{ab}{}_c(x, y; z) r^c_\beta(z, w).\end{aligned} \tag{3.49}$$

The action of $\mathcal{J}_X$ on the antifields is fixed by demanding that $[\mathcal{J}_X, Q_{KT}] = 0$. We find

$$\begin{aligned}
[\mathcal{J}_X, \phi_\alpha(x)] &= \int d^N y\ X_a(y) r^a_\alpha(y, x), \\
[\mathcal{J}_X, \phi^{*\alpha}(x)] &= -\iint d^N y\ d^N z\ (-)^{\alpha\beta+\beta} X_a(y)\phi^{*\beta}(z)\frac{\delta r^a_\beta(y, z)}{\delta\phi_\alpha(x)}, \\
[\mathcal{J}_X, \mathfrak{b}^a(x)] &= -\iint d^N y\ d^N z\ f^{ab}{}_c(x, y; z) X_b(y)\mathfrak{b}^c(z),
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
[\mathcal{J}_X, \pi^\alpha(x)] &= -\iint d^N y\ d^N z\ X_a(y)\frac{\delta r^a_\beta(y, z)}{\delta\phi_\alpha(x)}\pi^\beta(z), \\
[\mathcal{J}_X, \pi^*_\alpha(x)] &= \iint d^N y\ d^N z\ (-)^{\alpha\beta+\alpha} X_a(y)\frac{\delta r^a_\alpha(y, x)}{\delta\phi_\beta(z)}\pi^*_\beta(z), \\
[\mathcal{J}_X, \mathfrak{c}_a(x)] &= \iint d^N y\ d^N z\ f^{cb}{}_a(z, y; x) X_b(y)\mathfrak{c}_c(z).
\end{aligned} \tag{3.51}$$

The total generators are thus $\mathcal{J}^{\mathrm{TOT}}_X = \mathcal{J}_X + \mathcal{J}^{(1)}_X + \mathcal{J}^{(2)}_X$, where

$$\begin{aligned}
\mathcal{J}^{(1)}_X &= -\iiint d^Nx\ d^Ny\ d^Nz\ (-)^{\alpha\beta+\beta} X_a(y)\phi^{*\beta}(z)\frac{\delta r^a_\beta(y,z)}{\delta\phi_\alpha(x)}\pi^*_\alpha(x), \\
\mathcal{J}^{(2)}_X &= -\iiint d^Nx\ d^Ny\ d^Nz\ f^{ab}{}_c(x,y;z)X_b(y)\mathfrak{b}^c(z)\mathfrak{c}_a(x).
\end{aligned} \tag{3.52}$$

The Noether identity (3.39) can be rewritten as

$$\int d^Nx\ [\mathcal{J}_X, \phi_\alpha(x)]\mathcal{E}^\alpha(x) \equiv 0. \tag{3.53}$$

Hence not only do Noether identities imply local symmetries, but the converse is also true. Note that the bosonic character of the Noether identities is manifest here. In particular, diffeomorphism symmetry implies

$$\int d^Nx\ [\mathcal{L}_\xi, \phi_\alpha(x)]\mathcal{E}^\alpha(x) + \int dt\ [\mathcal{L}_\xi, q^\mu(t)]\mathcal{G}_\mu(t) = 0 \tag{3.54}$$

($diff(N)$ acts trivially on the einbein), and reparametrization symmetry gives

$$\int dt\ [L_f, q^\mu(t)]\mathcal{G}_\mu(t) + \int dt\ [L_f, e(t)]\mathcal{O}(t) \equiv 0 \tag{3.55}$$

($diff(1)$ acts trivially on the fields). The corresponding additions to the KT generator are

$$\begin{aligned}
Q^{(\mathrm{diff})}_{KT} &= \int d^Nx\ \Big(\int d^Ny\ (-)^\alpha [\mathcal{L}_\mu(x), \phi_\alpha(y)]\phi^{*\alpha}(y) + \\
&\quad + \int dt\ [\mathcal{L}_\mu(x), q^\nu(t)]q^*_\nu(t)\Big)\mathfrak{c}^\mu(x), \\
Q^{(\mathrm{rep})}_{KT} &= \iint dsdt\ ([L(s), q^\mu(t)]q^*_\mu(t) + [L(s), e(t)]e^*(t))\mathfrak{c}(s), \\
Q^{(\mathrm{gauge})}_{KT} &= \iint d^Nx\ d^Ny\ (-)^\alpha [\mathcal{J}^a(x), \phi_\alpha(y)]\phi^{*\alpha}(y)\mathfrak{c}_a(x),
\end{aligned} \tag{3.56}$$

where the localized generators were defined in (2.6), the Noether antifields are $\mathfrak{b}_\mu(x)$, $\mathfrak{b}(t)$, $\mathfrak{b}^a(x)$, and their momenta are $\mathfrak{c}^\mu(x)$, $\mathfrak{c}(t)$ and $\mathfrak{c}_a(t)$, respectively.

The gauge algebra needs only be satisfied up to a KT exact term. For every choice of fermionic operator K_X, the modified generators $\mathcal{J}'_X = \mathcal{J}_X +$

$[Q_{KT}, K_X]$ satisfy the same algebra in cohomology as does the original $\mathcal{J}_X$, although the brackets on $\Omega^\bullet_{KT}$ acquires a correction:

$$[\mathcal{J}'_X, \mathcal{J}'_Y] = \mathcal{J}'_{[X,Y]} + [Q_{KT}, [\mathcal{J}_X, K_Y] - [\mathcal{J}_Y, K_X] - K_{[X,Y]}]. \tag{3.57}$$

However, this freedom will not be exploited further.

3.7 Continuity equation

It often happens that the fields can be split into two disjoint sets, $\phi_\alpha = (\varphi_i, \psi_A)$, such that the action takes the form $S = S_1[\varphi] + S_2[\varphi, \psi]$. Typically, φ_i is a metric or gauge field, and ψ_A denote matter fields. Moreover, we demand that the Noether symmetries commute with each piece separately, i.e. $[\mathcal{J}_X, S_1] = [\mathcal{J}_X, S_2] = 0$. Then there are two independent identities

$$\int d^N y\ r_i^a(x,y) \frac{\delta S_1}{\delta \varphi_i(y)} \equiv 0, \tag{3.58}$$

$$\int d^N y\ r_i^a(x,y) \frac{\delta S_2}{\delta \varphi_i(y)} + r_A^a(x,y) \frac{\delta S_2}{\delta \psi_A(y)} \equiv 0. \tag{3.59}$$

However, these are not separately proportional to the EL equations

$$\frac{\delta S_1}{\delta \varphi_i(x)} + \frac{\delta S_2}{\delta \varphi_i(x)} = 0, \qquad \frac{\delta S_2}{\delta \psi_A(x)} = 0, \tag{3.60}$$

so only the sum of (3.58) and (3.59) imposes restrictions on the EL equations, provided, of course, that S_2 depends non-trivially on φ_i. Combining (3.58) and the EL equations we find

$$\int d^N y\ r_i^a(x,y) \frac{\delta S_2}{\delta \varphi_i(y)} = 0 \tag{3.61}$$

on the stationary surface. This is the continuity equation. However, it is not an identity that holds off the stationary surface, and therefore it needs not be eliminated in cohomology.

3.8 Examples

3.8.1 Maxwell-Dirac

The fields ϕ_α consist of the bosonic gauge potential A_μ and two independent fermionic Dirac spinors ψ and $\bar\psi$. The spacetime points x and spinor

indices are suppressed in this example, and fermionic brackets are explicitly indicated by $\{\cdot,\cdot\}$. For brevity, we assume the metric to be flat Minkowski, and freely use this metric to raise and lower indices, and hence $diff(N)$ is broken down to the Poincaré algebra. Let γ_μ denote gamma matrices, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Henceforth, we focus on the $map(N, u(1))$ Noether symmetry.

The action reads

$$S = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} + \int \bar\psi(\gamma^\mu(\partial_\mu + iA_\mu) - m)\psi, \tag{3.62}$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. According to our prescription, we introduce antifields and momenta as follows.

Fields	ϕ_α	A_μ	ψ	$\bar\psi \sim \psi^\dagger\gamma^0$
Momenta	π^α	E^μ	$\pi \sim \psi^\dagger$	$\bar\pi \sim \bar\psi^\dagger$
Antifields	$\phi^{*\alpha}$	$A^{*\mu}$	$\psi^* \sim \psi^\dagger$	$\psi^* \sim \bar\psi^\dagger$
Antifield momenta	π^*_α	E^*_μ	$\pi^* \sim \psi$	$\bar\pi^* \sim \bar\psi$,

where $\sim$ indicates the transformation properties under rotations. The nonzero Poisson brackets are

$$\begin{aligned} [E^\mu, A_\nu] &= \delta^\mu_\nu, & \{\pi, \psi\} &= \{\bar\pi, \bar\psi\} = 1, \\ \{E^*_\mu, A^{*\nu}\} &= \delta^\nu_\mu, & [\pi^*, \psi^*] &= [\bar\pi^*, \bar\psi^*] = 1. \end{aligned} \tag{3.63}$$

The EL equations are

$$\begin{aligned} \frac{\delta S}{\delta A_\mu} &\equiv [E^\mu, S] = \partial_\nu F^{\nu\mu} - j^\mu, \\ j^\mu &\equiv \bar\psi\gamma^\mu\psi, \\ \frac{\delta S}{\delta\psi} &\equiv [\pi, S] = (i\partial_\mu + A_\mu)\bar\psi\gamma^\mu + m\bar\psi, \\ \frac{\delta S}{\delta\bar\psi} &= [\bar\pi, S] = \gamma^\mu(i\partial_\mu - A_\mu)\psi - m\psi. \end{aligned} \tag{3.64}$$

The first part of the KT generator is

$$\begin{aligned} Q^{(1)}_{KT} \equiv& \int \mathcal{E}^\alpha\pi^*_\alpha = \int(\partial_\nu F^{\nu\mu} - j^\mu)E^*_\mu + \\ &+((i\partial_\mu + A_\mu)\bar\psi\gamma^\mu + m\bar\psi)\pi^* - \bar\pi^*(\gamma^\mu(i\partial_\mu - A_\mu)\psi - m\psi). \end{aligned} \tag{3.65}$$

The Noether identity reads

$$-\partial_\mu \frac{\delta S}{\delta A_\mu} + i\bar{\psi}\frac{\delta S}{\delta \bar{\psi}} + i\frac{\delta S}{\delta \psi}\psi = -\partial_\mu(\partial_\nu F^{\nu\mu} - j^\mu) + \tag{3.66}$$
$$+i\bar{\psi}(\gamma^\mu(i\partial_\mu - A_\mu)\psi - m\psi) + i((i\partial_\mu + A_\mu)\bar{\psi}\gamma^\mu + m\bar{\psi})\psi \equiv 0.$$

The corresponding gauge symmetry is $map(N, u(1))$, which acts as follows on the fields

$$[\mathcal{J}_X, A_\mu] = \partial_\mu X, \qquad [\mathcal{J}_X, \psi] = -iX\psi, \qquad [\mathcal{J}_X, \bar{\psi}] = iX\bar{\psi}. \tag{3.67}$$

To eliminate this symmetry, we must introduce the Noether antifield $\mathfrak{b}$, with momentum $\mathfrak{c}$: $[\mathfrak{c}, \mathfrak{b}] = 1$. They transform in the adjoint representation of the gauge algebra, which in this case is trivial since $u(1)$ is abelian: $[\mathcal{J}_X, \mathfrak{b}] = [\mathcal{J}_X, \mathfrak{c}] = 0$. The total gauge generator is

$$\mathcal{J}_X = \int \partial_\mu X E^\mu + iX(\pi\psi + \bar{\psi}\pi + \psi^*\pi^* + \bar{\pi}^*\bar{\psi}^*), \tag{3.68}$$

and the Noether contribution to the KT generator is

$$Q^{(2)}_{KT} = -\int (\partial_\mu A^{*\mu} + i\psi^*\psi + i\bar{\psi}\bar{\psi}^*)\mathfrak{c}. \tag{3.69}$$

In fact, (3.66) is of the form discussed in subsection (3.7). The first Noether identity (3.58) reads $\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0$, leading to the continuity equation $\partial_\mu j^\mu = 0$.

3.8.2 Yang-Mills and spinors

The example in the previous subsection can be extended to the Yang-Mills case, by replacing the gauge group $u(1)$ by an arbitrary semi-simple Lie algebra $\mathfrak{g}$. The modifications are straightforward and are left to the reader.

To describe spinors in a $diff(N)$ invariant manner requires a vielbein formalism. This reduces to the Yang-Mills case with gauge group $so(N)_{spin}$, except that we can define a vielbein $e^I_\mu(x)$ with inverse $e^{I\mu}(x)$ ($I, J, \ldots$ denote $so(N)$ vector indices), such that the spin connection and the metric are auxiliary fields, satisfying

$$\omega^{IJ}_\mu(x) = e^{[I}_\nu(x)\partial_\mu e^{J]\nu}(x), \qquad g_{\mu\nu}(x) = e^I_\mu(x)e^I_\nu(x). \tag{3.70}$$

3.8.3 Einstein

The action reads

$$S = S^{(E)}[g] + S'[g,\phi] + S^{(q)}[q,e,g], \tag{3.71}$$

where the Einstein action is

$$S^{(E)}[g] = \frac{1}{16\pi}\int d^Nx\ \sqrt{|g(x)|}\,R(x), \tag{3.72}$$

and $R(x)$ is the scalar curvature. Further, $S'[g,\phi]$ is the part of the action depending on other fields and $S^{(q)}[q,e,g]$ was defined in (3.32). The EL equation reads

$$\begin{aligned}\frac{\delta S}{\delta g_{\mu\nu}(x)} &= -\frac{1}{16\pi}\sqrt{|g(x)|}\Big(G^{\mu\nu}(x) - 8\pi T^{\mu\nu}(x) - \\ &\quad -8\pi\int dt\ e^{-1}(t)\dot q^\mu(t)\dot q^\nu(t)\delta^N(x-q(t))\Big) = 0,\end{aligned} \tag{3.73}$$

where $G^{\mu\nu} = R^{\mu\nu} - (1/2)g^{\mu\nu}R$ is the Einstein tensor and $T^{\mu\nu} = (2/\sqrt{|g|})$ $\delta S'/\delta g_{\mu\nu}$ is the energy-momentum tensor. The last, non-standard, term in (3.73) describes how the massive observer curves spacetime around herself.

Let $\pi^{\mu\nu}(x)$ denote the momentum conjugate to $g_{\mu\nu}(x)$, and let $g^{*\mu\nu}(x)$ and $\pi^*_{\mu\nu}(x)$ be the fermionic antifield and its momentum. The contribution to the KT generator is

$$\begin{aligned}Q^{(E)}_{KT} &= -\frac{1}{16\pi}\int d^Nx\ \sqrt{|g(x)|}\,(G^{\mu\nu}(x) - 8\pi T^{\mu\nu}(x))\pi^*_{\mu\nu}(x) + \\ &\quad +\frac{1}{2}\int dt\ \sqrt{|g(q(t))|}\,e^{-1}(t)\dot q^\mu(t)\dot q^\nu(t)\pi^*_{\mu\nu}(q(t)).\end{aligned} \tag{3.74}$$

The Noether symmetry $diff(N)$ is of the form discussed in subsection (3.7). The full identity depends on all fields, but there is also the identity $\partial_\nu G^{\mu\nu}(x) \equiv 0$, which leads to the continuity equation $\partial_\nu T^{\mu\nu}(x) = 0$.

3.8.4 Geodesic constraint

Continues subsection (3.4). The $diff(1)$ identity (3.55) becomes

$$\dot q^\mu(t)\mathcal{G}_\mu(t) - e(t)\dot{\mathcal{O}}(t) \equiv 0, \tag{3.75}$$

and the reparametrization contribution to the KT generator becomes

$$Q^{(\mathrm{rep})}_{KT} = \int dt\ (\dot q^\mu(t)q^*_\mu(t) - e(t)\dot e^*(t))\mathfrak{c}(t). \tag{3.76}$$

3.9 Comparison with the Batalin-Vilkovisky formalism

Since the formulation of classical physics that has been developped in the previous subsections is new, it makes sense to compare it with other approaches. The closest resemblance is with the antifield formalism of Batalin-Vilkovisky (BV), particularly in the cohomological formulation of [9, 16]. Similarly to these authors, I impose the EL equation in the cohomology generated by the KT differential. However, there are three major differences.

1. In the BV approach one considers a BRST complex rather than the KT complex, i.e. Noether symmetries are eliminated by the introduction of ghosts. This could be done in the present formalism as well. For each Noether identity (3.39), introduce a fermionic ghost $C_a(x)$ and a ghost momentum (or antighost) $B^a(x)$. The BRST generator

$$\begin{aligned} Q_{BRST} &= \int d^Nx\ C_a(x)\mathcal{J}^a(x) + \\ &+\frac{1}{2}\iiint d^Nx\ d^Ny\ d^Nz\ f^{ab}{}_c(x,y;z)C_a(x)C_b(y)B^c(z) \end{aligned} \tag{3.77}$$

is nilpotent, and its cohomology identifies points on the same gauge orbits. The total generator $Q_{TOT} = Q_{KT} + Q_{BRST}$ is also nilpotent, and its cohomology consists of gauge-equivalence classes of differential forms on the stationary surface.

However, the BRST generator will not appear in this work. Classically, this is a matter of taste; it is equivalent to view a space as a $\mathfrak{g}$ module or to consider its equivalence classes under the $\mathfrak{g}$ action. However, quantization will in general introduce abelian extensions ("anomalies"), which ruin the nilpotency of the BRST generator. Therefore, we only consider the KT generator, which is not affected by anomalies.

2. Not only do I use fields and antifields, but also field and antifield momenta. There is thus already a graded Poisson structure, in terms of which an antibracket (a non-zero fermionic bracket between the fields and antifields) can be defined. For any $f, g \in C^\infty(\mathcal{Q}) \otimes \mathbb{C}[\phi^*, \mathfrak{b}, C]$, set

$$\begin{aligned} (f,g) &= \int d^Nx \Big(-(-)^\alpha([f,\pi^*_\alpha(x)][\pi^\alpha(x),g] + [f,\pi^\alpha(x)][\pi^*_\alpha(x),g]) + \\ &\quad +[f,B^a(x)][\mathfrak{c}_a(x),g] + [f,\mathfrak{c}_a(x)][B^a(x),g]\Big) \\ &= -(-)^{(f+1)(g+1)}(g,f). \end{aligned} \tag{3.78}$$

In particular,

$$(\phi_\alpha(x), \phi^{*\beta}(y)) = \delta^\beta_\alpha \delta^N(x-y), \qquad (C_a(x), \mathfrak{b}^b(y)) = \delta^b_a \delta^N(x-y). \quad (3.79)$$

The KT differential on $C^\infty(\mathcal{Q}) \otimes \mathbb{C}[\phi^*, \mathfrak{b}, C]$ is now reproduced by $\delta f = (f, S_{\mathrm{TOT}})$, where

$$S_{\mathrm{TOT}} = S + \int d^N x \; d^N y \; \phi^{*\alpha}(y) r^a_\alpha(x,y) C_a(x) \qquad (3.80)$$

is the total action. Nilpotency leads to the classical master equation $(S_{\mathrm{TOT}}, S_{\mathrm{TOT}}) = 0$. However, this definition of δ can not be extended to all of $C^\infty(\mathcal{P}) \otimes \mathbb{C}[\phi^*, \pi^*, \mathfrak{b}, \mathfrak{c}, C, B]$, because $(\pi^\alpha(x), S_{\mathrm{TOT}}) = (\pi^*_\alpha(x), S_{\mathrm{TOT}}) = (B^a(x), S_{\mathrm{TOT}}) = 0$. Hence in the BV formalism, the KT complex only gives a resolution of the space of functions on the solution surface, whereas the ℓ-form spaces $H^\ell_\ell(Q_{KT})$ can only be resolved using the more general expression (3.12).

3. Momenta and velocitites are treated independently. Usually, they are identified by the equation

$$\pi^\alpha(x) = \frac{\partial(\sqrt{|g|(x)}\mathcal{L}(x;\phi))}{\partial \partial_0 \phi(x)}. \qquad (3.81)$$

This equation can be thought of as an extra constraint, from which either $\pi^\alpha(x)$ or $\partial_0\phi_\alpha(x)$ can be eliminated. However, this additional condition gives rise to three significant problems: First, it is not generally covariant, so there is little hope to represent $DRO(N)$ on the factor space. Second, it is a second class constraint, which can not be separated into a first class constraint and a gauge fixation in a natural way. Hence cohomological methods fail. Third, it can not be formulated in jet space, since $\pi^\alpha(x)$ can not be expanded in a Taylor series around $q(t)$. In view of these difficulties, velocities and momenta are kept as independent objects. The price for this seems modest: the cohomology groups contain the ℓ-form spaces for non-zero ℓ.

3.10 Quantization

Having formulated classical physics as the cohomology of the KT complex, we could now try to quantize it by reinterpreting the Poisson brackets (3.3) as commutators. The strategy is thus first to quantize and then recover the dynamics in cohomology. However, this leads to three major difficulties.

1. The geodesic equation contains the metric at the observer's present position, $g_{\mu\nu}(q(t))$. It is not clear what to do with this object if both $g_{\mu\nu}(x)$ and $q^\mu(t)$ are turned into operators.

2. There is no invariant time choice. Of course, we could make a Fourier transformation w.r.t. x^0, and define the Fock vacuum to be annihilated by negative energy modes, but such a decomposition is not invariant. Therefore, it is not clear that the Fock space carries a $diff(N)$ representation.

3. Normal ordering of the generators (3.4) is ill defined. More precisely, central extensions proportional to the number of x^0-independent functions arise, but this number is infinite except in one dimension.

These difficulties disappear if we expand the fields in a Taylor series around the observer's present position.

4 Jet space quantization

4.1 Jet space trajectories

Let $\mathbf{m} = (m_0, m_1, .., m_{N-1})$, all $m_\mu \geqslant 0$, be a multi-index of order $|\mathbf{m}| = \sum_{\mu=0}^{N-1} m_\mu$, let $\underline{\mu}$ be a unit vector in the μ:th direction, and let 0 be the empty multi-index of order zero. Expand $\phi_\alpha(x)$ in a power series around $q^\mu(t)$.

$$\phi_\alpha(x) = \sum_{|\mathbf{m}|\geqslant 0} \frac{1}{\mathbf{m}!} \phi_{\alpha,\mathbf{m}}(t)(x - q(t))^{\mathbf{m}}, \tag{4.1}$$

where $\mathbf{m}! = m_0! m_1! .. m_{N-1}!$ and

$$(x - q(t))^{\mathbf{m}} = (x^0 - q^0(t))^{m_0}(x^1 - q^1(t))^{m_1} .. (x^{N-1} - q^{N-1}(t))^{m_{N-1}}. \tag{4.2}$$

Since the DGRO algebra acts on $C^\infty(\mathcal{Q})$, it also acts on the infinite jet space $J^\infty\mathcal{Q}$, with basis $(\phi_{\alpha,\mathbf{m}}(t), q^\mu(t), e(t))$, $t \in S^1$. The transformation law is described in (4.8) below. $DGRO(N, \mathfrak{g})$ also acts on the p-jet spaces $J^p\mathcal{Q}$, p finite, obtained by truncating to $|\mathbf{m}| \leqslant p$. The realization on $J^p\mathcal{Q}$ is non-linear in the trajectory $q^\mu(t)$, so it must be interpreted as a linear representation on $C^\infty(J^p\mathcal{Q})$, or more restrictively as a representation on $J^p\mathcal{Q}\otimes_{[q(t)]} Obs(N)$, where the observer algebra $Obs(N)$ was defined in section 2 and $q^\mu(t)$ is identified in both factors.

The corresponding phase space $J^p\mathcal{P}$ is obtained by adjoining to $J^p\mathcal{Q}$ dual coordinates $(\pi^{\alpha,\mathbf{m}}(t), p_\mu(t), \pi_e(t))$. The only non-zero brackets are

$$\begin{aligned}
[p_\mu(s), q^\nu(t)] &= \delta^\nu_\mu \delta(s-t), \\
[\pi_e(s), e(t)] &= \delta(s-t), \\
[\pi^{\alpha,\mathbf{m}}(s), \phi_{\beta,\mathbf{n}}(t)] &\equiv -(-)^{\alpha\beta}[\phi_{\beta,\mathbf{n}}(t), \pi^{\alpha,\mathbf{m}}(s)] = \delta^\alpha_\beta \delta^{\mathbf{m}}_{\mathbf{n}} \delta(s-t).
\end{aligned} \tag{4.3}$$

Observe that the $\pi^{\alpha,\mathbf{m}}(t)$ are not the Taylor coefficients of $\pi^\alpha(x)$, because the latter can not be expanded in a power series in $(x-q(t))$. The reason is that $\delta^N(x-y)$ can not be written as a double power series. The delta function does have the expansion

$$\delta^N(x-y) = \sum_{\mathbf{m}\in\mathbb{Z}} (x-q(t))^{\mathbf{m}}(y-q(t))^{-\mathbf{m}-\underline{1}}, \tag{4.4}$$

where $\underline{1} = (1,1,\dots,1)$, but to use this expression in (3.3), $\phi_\alpha(x)$ and $\pi^\alpha(x)$ must be expanded in formal Laurent (rather than power) series. Since such an assumption leads to the type of infinities that we want to avoid, we simply postulate no relation between $\pi^\alpha(x)$ and $\pi^{\alpha,\mathbf{m}}(t)$. It is now clear why (3.81) has no jet space analogue: $\pi^{\alpha,\mathbf{m}}(t)$ has an upper multi-index whereas any function of $\phi_{\alpha,\mathbf{m}}(t)$ can only have multi-indices downstairs.

Define $T^{\mathbf{m}}_{\mathbf{n}}(\xi), J^{\mathbf{m}}_{\mathbf{n}}(X) \in \mathcal{U}(gl(N)\oplus\mathfrak{g})$ (universal envelopping algebra) by

$$\begin{aligned} T^{\mathbf{m}}_{\mathbf{n}}(\xi) &= \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n}-\mathbf{m}+\underline{\nu}}\xi^\mu T^\nu_\mu \\ &+ \binom{\mathbf{n}}{\mathbf{m}-\underline{\mu}}(1-\delta^{\mathbf{m}-\underline{\mu}}_{\mathbf{n}})\partial_{\mathbf{n}-\mathbf{m}+\underline{\mu}}\xi^\mu I, \\ J^{\mathbf{m}}_{\mathbf{n}}(X) &= \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n}-\mathbf{m}} X_a J^a. \end{aligned} \tag{4.5}$$

where $\binom{\mathbf{n}}{\mathbf{m}} = \mathbf{n}!/\mathbf{m}!(\mathbf{n}-\mathbf{m})!$ and I is the unit element in $\mathcal{U}(gl(N)\oplus\mathfrak{g})$. These objects satisfy

$$\begin{aligned} T^{\mathbf{m}}_{\mathbf{n}+\underline{\nu}}(\xi) &= \partial_\nu\xi^\mu\delta^{\mathbf{m}}_{\mathbf{n}+\underline{\mu}} I + T^{\mathbf{m}}_{\mathbf{n}}(\partial_\nu\xi) + T^{\mathbf{m}-\underline{\nu}}_{\mathbf{n}}(\xi), \\ T^{\mathbf{m}}_0(\xi) &= \delta^{\mathbf{m}}_0 \partial_\nu\xi^\mu T^\nu_\mu, \\ \partial_\nu T^{\mathbf{m}}_{\mathbf{n}}(\xi) &= T^{\mathbf{m}}_{\mathbf{n}}(\partial_\nu\xi), \\ T^{\mathbf{m}}_{\mathbf{n}}([\xi,\eta]) &= \xi^\mu T^{\mathbf{m}}_{\mathbf{n}}(\partial_\mu\eta) - \eta^\nu T^{\mathbf{m}}_{\mathbf{n}}(\partial_\nu\xi) \\ &+ \sum_{|\mathbf{m}|\leqslant|\mathbf{r}|\leqslant|\mathbf{n}|} T^{\mathbf{r}}_{\mathbf{n}}(\xi)T^{\mathbf{m}}_{\mathbf{r}}(\eta) - T^{\mathbf{r}}_{\mathbf{n}}(\eta)T^{\mathbf{m}}_{\mathbf{r}}(\xi), \end{aligned} \tag{4.6}$$

$$\begin{aligned} J^{\mathbf{m}}_{\mathbf{n}+\underline{\mu}}(X) &= J^{\mathbf{m}}_{\mathbf{n}}(\partial_\mu X) + J^{\mathbf{m}-\underline{\mu}}_{\mathbf{n}}(X), \\ J^{\mathbf{m}}_0(X) &= \delta^{\mathbf{m}}_0 X_a J^a, \\ \partial_\mu J^{\mathbf{m}}_{\mathbf{n}}(X) &= J^{\mathbf{m}}_{\mathbf{n}}(\partial_\mu X), \\ J^{\mathbf{m}}_{\mathbf{n}}([X,Y]) &= \sum_{|\mathbf{m}|\leqslant|\mathbf{r}|\leqslant|\mathbf{n}|} J^{\mathbf{r}}_{\mathbf{n}}(X)J^{\mathbf{m}}_{\mathbf{r}}(Y) - J^{\mathbf{r}}_{\mathbf{n}}(Y)J^{\mathbf{m}}_{\mathbf{r}}(X), \end{aligned} \tag{4.7}$$

$$J^{\mathbf{m}}_{\mathbf{n}}(\xi^\mu \partial_\mu X) \;=\; \xi^\mu J^{\mathbf{m}}_{\mathbf{n}}(\partial_\mu X) + \sum_{|\mathbf{m}|\leqslant|\mathbf{r}|\leqslant|\mathbf{n}|} T^{\mathbf{r}}_{\mathbf{n}}(\xi) J^{\mathbf{m}}_{\mathbf{r}}(X) - J^{\mathbf{r}}_{\mathbf{n}}(X) T^{\mathbf{m}}_{\mathbf{r}}(\xi).$$

In particular, $T^{\mathbf{m}}_{\mathbf{n}}(\xi) = J^{\mathbf{m}}_{\mathbf{n}}(X) = 0$ if $|\mathbf{m}| > |\mathbf{n}|$. Alternatively, (4.6–4.7) could be taken as a recursive definition of $T^{\mathbf{m}}_{\mathbf{n}}(\xi)$ and $J^{\mathbf{m}}_{\mathbf{n}}(X)$. Every $gl(N)\oplus\mathfrak{g}$ representation ϱ on V clearly gives a representation of these operators.

We can now write down the $DGRO(N,\mathfrak{g})$ action on $J^p\mathcal{P}$.

$$\begin{aligned}
[\mathcal{L}_\xi, \phi_{\alpha,\mathbf{n}}(t)] &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|} \varrho^\beta_\alpha(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t))))\phi_{\beta,\mathbf{m}}(t), \\
[\mathcal{J}_X, \phi_{\alpha,\mathbf{n}}(t)] &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|} \varrho^\beta_\alpha(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t))))\phi_{\beta,\mathbf{m}}(t), \\
[L_f, \phi_{\alpha,\mathbf{n}}(t)] &= -f(t)\dot\phi_{\alpha,\mathbf{n}}(t) - \lambda \dot f(t)\phi_{\alpha,\mathbf{n}}(t),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[\mathcal{J}_X, q^\mu(t)] &= 0, \\
[L_f, q^\mu(t)] &= -f(t)\dot q^\mu(t),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
[\mathcal{L}_\xi, e(t)] &= [\mathcal{J}_X, e(t)] = 0, \\
[L_f, e(t)] &= -f(t)\dot e(t) - \dot f(t) e(t),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
[\mathcal{L}_\xi, \pi^{\alpha,\mathbf{m}}(t)] &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p} \pi^{\beta,\mathbf{n}}(t)\varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t)))), \\
[\mathcal{J}_X, \pi^{\alpha,\mathbf{m}}(t)] &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p} \pi^{\beta,\mathbf{n}}(t)\varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t)))), \\
[L_f, \pi^{\alpha,\mathbf{m}}(t)] &= -f(t)\dot\pi^{\alpha,\mathbf{m}}(t) - (1-\lambda)\dot f(t)\pi^{\alpha,\mathbf{m}}(t),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
[\mathcal{L}_\xi, p_\nu(t)] &= -\partial_\nu\xi^\mu(q(t))p_\mu(t) + \\
&\quad + \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p} \varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\partial_\nu\xi(q(t))))\phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}(t), \\
[\mathcal{J}_X, p_\nu(t)] &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p} \varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(\partial_\nu X(q(t))))\phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}(t), \\
[L_f, p_\nu(t)] &= -f(t)\dot p_\nu(t) - \dot f(t)p_\nu(t),
\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
[\mathcal{L}_\xi, \pi_e(t)] &= [\mathcal{J}_X, \pi_e(t)] = 0, \\
[L_f, \pi_e(t)] &= -f(t)\dot{\pi}_e(t).
\end{aligned}
\tag{4.13}
$$

Actually, this transformation law is more general than what follows from (4.1), because we have included an extra term proportional to the parameter λ in (4.8), without spoiling the representation condition. We call λ the *causal weight*, and note that the einbein $e(t)$ is a zero-jet in the trivial $gl(N) \oplus \mathfrak{g}$ representation with unit causal weight.

The observer's trajectory does not transform as a zero-jet, but its derivative does (with causal weight one):

$$
\begin{aligned}
[\mathcal{L}_\xi, \dot{q}^\mu(t)] &= \partial_\nu \xi^\mu(q(t))\dot{q}^\nu(t), \\
[\mathcal{J}_X, \dot{q}^\mu(t)] &= 0, \\
[L_f, \dot{q}^\mu(t)] &= -f(t)\ddot{q}^\mu(t) - \dot{f}(t)\dot{q}^\mu(t).
\end{aligned}
\tag{4.14}
$$

In view of the field-dependent terms in (4.12), $p_\mu(t)$ no longer transforms in a simple fashion under $diff(N)$. However,

$$
P_\mu(t) = p_\mu(t) + \sum_{\mathbf{m}} \phi_{\alpha,\mathbf{m}+\underline{\mu}}(t)\pi^{\alpha,\mathbf{m}}(t),
\tag{4.15}
$$

has a simple transformation law in the infinite jet space $J^\infty\mathcal{P}$. This formula suggests that we define the total derivative as

$$
\check{\partial}_\mu f = \int dt\ [P_\mu(t), f], \qquad \forall f \in C^\infty(J^\infty\mathcal{Q}).
\tag{4.16}
$$

The name is motivated by the following formulas:

$$
\check{\partial}_\mu q^\nu(t) = \delta^\nu_\mu, \qquad \check{\partial}_\mu \phi_{\alpha,\mathbf{n}}(t) = \phi_{\alpha,\mathbf{n}+\underline{\mu}}(t).
\tag{4.17}
$$

In the finite jet case, the total derivative is a map $\check{\partial}_\mu : C^\infty(J^p\mathcal{Q}) \longrightarrow C^\infty(J^{p+1}\mathcal{Q})$.

4.2 Fock space and normal ordering

All functions over S^1 can be expanded in a Fourier series, e.g.,

$$
\begin{aligned}
\phi_{\alpha,\mathbf{m}}(t) &= \sum_{r=-\infty}^{\infty} \hat{\phi}_{\alpha,\mathbf{m}}(r)\mathrm{e}^{-irt} \equiv \phi^<_{\alpha,\mathbf{m}}(t) + \phi^\geqslant_{\alpha,\mathbf{m}}(t), \\
\pi^{\alpha,\mathbf{m}}(t) &= \sum_{r=-\infty}^{\infty} \hat{\pi}^{\alpha,\mathbf{m}}(r)\mathrm{e}^{-irt} \equiv \pi^{\alpha,\mathbf{m}}_\leqslant(t) + \pi^{\alpha,\mathbf{m}}_>(t),
\end{aligned}
\tag{4.18}
$$

where the sums in $(\phi^{<}_{\alpha,\mathbf{m}}(t), \phi^{\geqslant}_{\alpha,\mathbf{m}}(t), \pi^{\alpha,\mathbf{m}}_{\leqslant}(t), \pi^{\alpha,\mathbf{m}}_{>}(t))$ are taken over (negative, non-negative, non-positive, positive) frequency modes only. Similarly, $q^\mu(t)$, $p_\nu(t)$, $e(t)$ and $\pi_e(t)$ are divided into positive and negative frequency modes.

Quantization amounts to replacing the Poisson brackets (4.3) by graded commutators; the Fock space $J^p\mathcal{F}$ is the universal envelopping algebra of (4.3) modulo relations

$$q^\mu_<(t)|0\rangle = p^{\leqslant}_\mu(t)|0\rangle = \phi^{<}_{\alpha,\mathbf{m}}(t)|0\rangle = \pi^{\alpha,\mathbf{m}}_{\leqslant}(t)|0\rangle = e_<(t)|0\rangle = \pi^{\leqslant}_e(t)|0\rangle = 0. \tag{4.19}$$

The dual Fock space $J^p\mathcal{F}'$ is built from a dual vacuum $\langle 0|$, annihilated by the remaining operators.

$$\langle 0|q^\mu_{\geqslant}(t) = \langle 0|p^{>}_\mu(t) = \langle 0|\phi^{\geqslant}_{\alpha,\mathbf{m}}(t) = \langle 0|\pi^{\alpha,\mathbf{m}}_{>} = \langle 0|e_{\geqslant}(t) = \langle 0|\pi^{>}_e(t). \tag{4.20}$$

Eqs. (4.19) and (4.20) together imply that the vacuum expectation value $\langle 0|\phi_{\alpha,\mathbf{m}}(t)|0\rangle = 0$ for every $\phi_{\alpha,\mathbf{m}}(t)$. This is not consistent with the condition that the metric and einbein have inverses. Therefore we define $g_{\mu\nu,\mathbf{m}}(t) = \eta_{\mu\nu}\delta_\mathbf{m} + h_{\mu\nu,\mathbf{m}}(t)$ and $e(t) = 1 + e'(t)$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, and rather demand that $h_{\mu\nu,\mathbf{m}}(t)$ and $e'(t)$ satisfy (4.19) and (4.20). Note that this decomposition is quite general; $h_{\mu\nu,\mathbf{m}}(t)$ is not required to be small, only to have vanishing vacuum expectation value. Moreover, the geodesic equation in vacuum reads $\ddot{q}^\mu(t) = 0$, so $q^\mu(t)$ may contain a linear part with non-zero vacuum expectation value.

The Fock spaces $J^p\mathcal{F}$ and $J^p\mathcal{F}'$ are *not* isomorphic.

Normal ordering is necessary to remove infinites and to obtain a well defined action on the Fock space. For every $F(q,e,\phi) \in C^\infty(J^p\mathcal{Q})$ (independent of all canonical momenta), denote

$$\begin{aligned} :F(q,e,\phi)p_\mu(t): &= F(q,e,\phi)p^{\leqslant}_\mu(t) + p^{>}_\mu(t)F(q,e,\phi), \\ :\phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}(t): &= \phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}_{\leqslant}(t) + (-)^{\alpha\beta}\pi^{\beta,\mathbf{n}}_{>}(t)\phi_{\alpha,\mathbf{m}}(t). \end{aligned} \tag{4.21}$$

4.3 Some definitions

Before describing the $DGRO(N)$ action on $J^p\mathcal{F}$, some more preparation is needed. Let $A = (A^\alpha_\beta)$ be a matrix acting on V. Its supertrace is $\mathrm{str}A = (-)^\alpha A^\alpha_\alpha = \sum_{\alpha \text{ bosonic}} A^\alpha_\alpha - \sum_{\alpha \text{ fermionic}} A^\alpha_\alpha$. For every $gl(N) \oplus \mathfrak{g}$ representation ϱ acting on V, define the numbers $\mathrm{sd}(\varrho)$ (super dimension), $k_0(\varrho)$, $k_1(\varrho)$, $k_2(\varrho)$, $y(\varrho)$, $z(\varrho)$, and $k_z(\varrho)$ by

$$\mathrm{str}(I) = \mathrm{sd}(\varrho),$$

$$
\begin{aligned}
\mathrm{str}(T^\mu_\nu) &= k_0(\varrho)\delta^\mu_\nu, \\
\mathrm{str}(T^\mu_\nu T^\sigma_\tau) &= k_1(\varrho)\delta^\mu_\tau\delta^\sigma_\nu + k_2(\varrho)\delta^\mu_\nu\delta^\sigma_\tau, \\
\mathrm{str}(J^a) &= z(\varrho)\delta^a, \\
\mathrm{str}(J^a J^b) &= y(\varrho)\delta^{ab} \\
\mathrm{str}(J^a T^\mu_\nu) &= k_z(\varrho)\delta^a\delta^\mu_\nu.
\end{aligned}
\tag{4.22}
$$

Since $gl(N) \cong sl(N) \oplus gl(1)$, its generators can be written as

$$
T^\mu_\nu = S^\mu_\nu + \omega\delta^\mu_\nu I,
\tag{4.23}
$$

where $S^\mu_\nu = T^\mu_\nu - (1/N)\delta^\mu_\nu T^\rho_\rho$ are the generators of $sl(N)$, $S^\mu_\mu = 0$, and $\mathrm{str}(S^\mu_\nu) = 0$. For our purposes, $sl(N) \cong A_{N-1} = su(N)$, and $\dim sl(N) = N^2 - 1$. The scalar ω, which labels the $gl(1)$ irreps, is related to the $gl(N)$ weight κ: $\omega = -\kappa + p - q$, where p (q) denotes the number of upper (lower) tensor indices. Every $gl(N) \oplus \mathfrak{g}$ representation can be written as $\varrho = \sum_{i\in I} R_i \oplus \omega_i \oplus M_i$, where R_i, ω_i and M_i denote irreducible $sl(N)$, $gl(1)$ and $\mathfrak{g}$ representations and I is some index set. To each irrep we assign a Grassmann parity factor $(-)^i$. The parameters in (4.22) can then be written

$$
\begin{aligned}
\mathrm{sd}(\varrho) &= \sum_{i\in I}(-)^i \dim R_i \dim M_i, \\
k_0(\varrho) &= \sum_{i\in I}(-)^i k_0(R_i,\omega_i) \dim M_i, \\
k_1(\varrho) &= \sum_{i\in I}(-)^i k_1(R_i) \dim M_i, \\
k_2(\varrho) &= \sum_{i\in I}(-)^i k_2(R_i,\omega_i) \dim M_i, \\
y(\varrho) &= \sum_{i\in I}(-)^i \dim R_i\, y(M_i), \\
z(\varrho) &= \sum_{i\in I}(-)^i \dim R_i\, z(M_i), \\
k_z(\varrho) &= \sum_{i\in I}(-)^i k_0(R_i,\omega_i)\, z(M_i).
\end{aligned}
\tag{4.24}
$$

For $R \oplus \omega$ a $gl(N)$ irrep,

$$
\begin{aligned}
k_0(R,\omega) &= \omega \dim R, \\
k_1(R) &= \frac{2x_R}{\dim R},
\end{aligned}
\tag{4.25}
$$

$$k_2(R,\omega) \quad = \quad \omega^2 \dim R - \frac{2x_R}{N \dim R},$$

where x_R is the Dynkin index (a positive integer) of the representation R; it is related to the value of the quadratic Casimir as $Q_R = 2x_R(N^2-1)/\dim R$. For $\mathfrak{g}$ semisimple,

$$z(M) = 0, \qquad y(M) = \frac{2\dim \mathfrak{g}}{\dim M} x_M, \tag{4.26}$$

where x_M is the Dynkin index of the representation M [7, 8].

The calculation of the abelian charges will use the following results [13].

$$\begin{aligned}
&\sum_{|\mathbf{m}|\leqslant p} \delta^{\mathbf{m}}_{\mathbf{m}} \operatorname{str}(I) = \binom{N+p}{p} \operatorname{sd}(\varrho), \\
&\sum_{|\mathbf{m}|\leqslant p} \operatorname{str}(T^{\mathbf{m}}_{\mathbf{m}}(\xi)) = \partial_\mu \xi^\mu \Big(\binom{N+p}{p} k_0(\varrho) - \binom{N+p}{p-1} \operatorname{sd}(\varrho) \Big), \\
&\sum_{|\mathbf{m}|,|\mathbf{n}|\leqslant p} \operatorname{str}(T^{\mathbf{m}}_{\mathbf{n}}(\xi) T^{\mathbf{n}}_{\mathbf{m}}(\eta)) = \partial_\nu \xi^\mu \partial_\mu \eta^\nu \Big(\binom{N+p}{p} k_1(\varrho) + \binom{N+p}{p-1} \operatorname{sd}(\varrho) \Big) + \\
&\quad + \partial_\mu \xi^\mu \partial_\nu \eta^\nu \Big(\binom{N+p}{p} k_2(\varrho) + \frac{N+1}{N} \binom{N+p}{p-2} \operatorname{sd}(\varrho) - 2 \binom{N+p}{p-1} k_0(\varrho) \Big), \\
&\sum_{|\mathbf{m}|\leqslant p} \operatorname{str}(J^{\mathbf{m}}_{\mathbf{m}}(X)) = X_a z(\varrho) \delta^a \binom{N+p}{p}, \qquad\qquad (4.27) \\
&\sum_{|\mathbf{m}|,|\mathbf{n}|\leqslant p} \operatorname{str}(J^{\mathbf{m}}_{\mathbf{n}}(X) J^{\mathbf{n}}_{\mathbf{m}}(Y)) = y(\varrho) \binom{N+p}{p} X_a Y_b \delta^{ab}, \\
&\sum_{|\mathbf{m}|,|\mathbf{n}|\leqslant p} \operatorname{str}(T^{\mathbf{m}}_{\mathbf{n}}(\xi) J^{\mathbf{n}}_{\mathbf{m}}(X)) = \\
&\quad = \partial_\mu \xi^\mu X_a \delta^a \Big(\binom{N+p}{p} k_z(\varrho) - \binom{N+p}{p-1} z(\varrho) \Big).
\end{aligned}$$

Compared to [13], a minus sign for fermions has been absorbed into the definition (4.24).

4.4 $DGRO(N)$ action on $J^p\mathcal{F}$

The main result of [13] is the explicit description of the DGRO algebra action on $J^p\mathcal{F}$. It follows from theorems 5.1 and 6.2 in that paper that the

following operators provide a realization of the $DGRO(N)$.

$$\begin{aligned}
\mathcal{L}_\xi &= \int dt\ :\xi^\mu(q(t))p_\mu(t): - \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p} \varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t)))) :\phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}(t): + \\
&\quad + \frac{u_1}{2\pi i}\int dt\ \partial_\mu\xi^\mu(q(t)), \\
\mathcal{J}_X &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p}\int dt\ \varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t)))) :\phi_{\alpha,\mathbf{m}}(t)\pi^{\beta,\mathbf{n}}(t): + \\
&\quad + \frac{u_2}{2\pi i}\delta^a\int dt\ X_a(q(t)), \qquad (4.28) \\
L_f &= \int dt\ f(t)(- :\dot q^\mu(t)p_\mu(t): - :\dot e(t)\pi_e(t): - \sum_{|\mathbf{m}|\leqslant p} :\dot\phi_{\alpha,\mathbf{m}}(t)\pi^{\alpha,\mathbf{m}}(t):) - \\
&\quad - \dot f(t)(:e(t)\pi_e(t): + \lambda\sum_{|\mathbf{m}|\leqslant p} :\phi_{\alpha,\mathbf{m}}(t)\pi^{\alpha,\mathbf{m}}(t):) + \frac{u_3}{2\pi i},
\end{aligned}$$

Compared to [13], the two terms proportional to u_1 and u_2 have been added, to fix normalization of the trivial cocycles in (2.2) and (2.5).

$$\begin{aligned}
u_1 &= -\lambda\Big(\binom{N+p}{p}k_0(\varrho) - \binom{N+p}{p-1}\,\mathrm{sd}(\varrho)\Big), \\
u_2 &= -\lambda\, z(\varrho)\binom{N+p}{p}, \qquad (4.29) \\
u_3 &= \frac{1}{2}(\lambda-\lambda^2)\,\mathrm{sd}(\varrho)\binom{N+p}{p},
\end{aligned}$$

which define funtions $u_j(p,N;\varrho,\lambda)$, $j=1,2,3$. The presence of these terms, as well as normal ordering, modifies the transformation law (4.12) for $p_\nu(t)$. The abelian charges,

$$c_j = c_j^{(q)}(N) + c_j^{(e)}(1) + c_j^{(\phi)}(p,N;\varrho,0), \qquad (4.30)$$

are given in terms of functions $c_j^{(q)}(N)$ (contribution from the observer's trajectory), $c_j^{(e)}(\lambda)$ (contribution from the einbein), and $c_j^{(\phi)}(p,N;\varrho,\lambda)$ (contribution from the fields).

$$\begin{aligned}
c_1^{(q)}(N) &= 1, \qquad c_2^{(q)}(N) = 0, \\
c_3^{(q)}(N) &= 1, \qquad c_4^{(q)}(N) = 2N, \qquad (4.31)
\end{aligned}$$

$$
\begin{aligned}
c_5^{(q)}(N) &= c_6^{(q)}(N) = c_7^{(q)}(N) = 0, \\
c_4^{(e)}(\lambda) &= 2(1-6\lambda+6\lambda^2), \qquad c_j^{(e)}(\lambda) = 0 \text{ otherwise},
\end{aligned} \tag{4.32}
$$

$$
\begin{aligned}
c_1^{(\phi)}(p,N;\varrho,\lambda) &= \binom{N+p}{p} k_1(\varrho) + \binom{N+p}{p-1} \operatorname{sd}(\varrho), \\
c_2^{(\phi)}(p,N;\varrho,\lambda) &= \binom{N+p}{p} k_2(\varrho) + \\
&+ \frac{N+1}{N}\binom{N+p}{p-2} \operatorname{sd}(\varrho) - 2\binom{N+p}{p-1} k_0(\varrho), \\
c_3^{(\phi)}(p,N;\varrho,\lambda) &= (2\lambda-1)\Big(\binom{N+p}{p} k_0(\varrho) - \binom{N+p}{p-1} \operatorname{sd}(\varrho)\Big), \\
c_4^{(\phi)}(p,N;\varrho,\lambda) &= 2(1-6\lambda+6\lambda^2)\binom{N+p}{p} \operatorname{sd}(\varrho), \\
c_5^{(\phi)}(p,N;\varrho,\lambda) &= -\binom{N+p}{p} y(\varrho), \\
c_6^{(\phi)}(p,N;\varrho,\lambda) &= (2\lambda-1)\binom{N+p}{p} z(\varrho), \\
c_7^{(\phi)}(p,N;\varrho,\lambda) &= \binom{N+p}{p-1} z(\varrho) - \binom{N+p}{p} k_z(\varrho).
\end{aligned} \tag{4.33}
$$

Define

$$
\tilde{c}_j^{(\phi)}(p,N;\varrho,\lambda) = c_j^{(\phi)}(p,N+1;\varrho,\lambda) - c_j^{(\phi)}(p-1,N+1;\varrho,\lambda). \tag{4.34}
$$

For future reference we record the formulas

$$
\begin{aligned}
\binom{N+p}{p} - \binom{N+p-1}{p-1} &= \binom{N-1+p}{p}, \\
\binom{N+p}{p-1} - \binom{N+p-1}{p-2} &= \binom{N-1+p}{p-1}, \\
\binom{N+p}{p-2} - \binom{N+p-1}{p-3} &= \binom{N-1+p}{p-2},
\end{aligned} \tag{4.35}
$$

which imply that $\tilde{c}_j^{(\phi)}(p,N;\varrho,\lambda) = c_j^{(\phi)}(p,N;\varrho,\lambda)$, if $j \neq 2$. An analogous formula holds for u_j:

$$
u_j(p,N;\varrho,\lambda) - u_j(p-1,N;\varrho,\lambda) = u_j(p,N-1;\varrho,\lambda). \tag{4.36}
$$

5 Constraints in jet space

5.1 State cohomology

The Fock spaces described in the previous sections are good quantum theories in the sense that they carry well-defined representations of $DGRO(N, \mathfrak{g})$, with a non-trivial abelian extension. However, neither are they reducible, nor is dynamics (EL equations) taken into account. The strategy for constructing smaller modules is to take the KT generator Q_{KT} from section 3, expand in a Taylor series in $(x - q(t))$, and discard all terms involving derivatives of order higher than p. Since the KT generator is invariant, the state cohomology defines a $DGRO(N, \mathfrak{g})$ module.

Define a physical state $|phys\rangle \in J^p\mathcal{F}$ as a state that is annihilated by the KT generator, $Q_{KT}|phys\rangle = 0$. The state cohomology $H^\bullet_{state} \equiv H^\bullet_{state}(Q_{KT}, J^p\mathcal{F})$ is the space of physical states modulo relations $|phys\rangle \sim |phys\rangle + Q_{KT}|\rangle$, i.e. the cohomology of the complex $\Omega^\bullet_{state} = \sum_{g=-\infty}^{\infty} \Omega^g_{state}$, where $\Omega^g_{state} = \{\Psi_g|0\rangle : \Psi_g \in J^p\mathcal{P}, \mathrm{gh}\,\Psi_g = g\}$. For $H^\bullet_{state}$ to be well defined, Q_{KT} must be normal ordered. However, Q_{KT} is always bilinear in commuting variables, so normal ordering has no effect. This is the crucial reason to prefer the KT generator over the BRST one.

The dual state cohomology $H^\bullet_{state}(Q_{KT}, J^p\mathcal{F}')$ is the space of dual physical states $\langle phys'| \in J^p\mathcal{F}'$, satisfying

$$\langle phys'|Q_{KT} = 0, \qquad \langle phys'| \sim \langle phys'| + \langle|Q_{KT}. \tag{5.1}$$

The two cohomologies are not isomorphic, since the underlying spaces $J^p\mathcal{F}$ and $J^p\mathcal{F}'$ are not so. However, two states of fixed ghosts number,

$$|g, phys\rangle = \Psi_g|0\rangle, \qquad \langle phys', g| = \langle 0|\Psi'_g, \tag{5.2}$$

where $\mathrm{gh}\,\Psi_g = \mathrm{gh}\,\Psi'_g = g$, satisfy orthogonality conditions of the form

$$\langle phys', g' \,|\, g, phys\rangle \propto \delta_{g+g'}. \tag{5.3}$$

In the remainder of this section we focus on the ket state cohomology $H^\bullet_{state}$ only.

A physical operator A_{phys} satisfies

$$[Q_{KT}, A_{phys}] = 0, \qquad A_{phys} \sim A_{phys} + [Q_{KT}, C]. \tag{5.4}$$

Only physical operators act in a well defined manner on $H^\bullet_{state}$, and hence we must demand that that all $DGRO(N, \mathfrak{g})$ generators are physical operators.

Note that normal ordering does not affect the conditions (5.4) if A_{phys} is at most linear in the momenta, because normal ordering has no effect on Q_{KT}.

The momentum number mom is no longer well defined, since operators of non-zero momentum number are created out of the vacuum. In the simplest case where there are neither fields nor einbein,

$$L_{\exp(imt)}|0\rangle = \int dt\ \mathrm{e}^{imt} p_\mu^>(t)\dot{q}^\mu(t)|0\rangle, \tag{5.5}$$

which is non-empty for positive m. Hence Ω^g_{state} can not be further decomposed into states of fixed momentum number, and all cohomology groups H^g_{state} are in general non-zero. However, physical states of non-zero ghost number decouple due to the orthogonality relation (5.3), and each cohomology group is a well-defined $DGRO(N,\mathfrak{g})$ module.

5.2 Longitudinal constraint

The Taylor coefficents depend on the parameter t although the field itself does not, because the expansion point $q^\mu(t)$ does. On the other hand, the RHS of (4.1) actually defines a function $\phi_\alpha(x,t)$ of two variables. To resolve this paradox we must impose the condition $\partial\phi_\alpha(x,t)/\partial t = 0$, which is equivalent to

$$\mathcal{D}_{\alpha,\mathbf{m}}(t) \equiv \dot{\phi}_{\alpha,\mathbf{m}}(t) - \dot{q}^\mu(t)\phi_{\alpha,m+\underline{\mu}}(t) \approx 0. \tag{5.6}$$

Introduce an antifield $\beta_{\alpha,\mathbf{m}}(t)$, with momentum $\gamma^{\alpha,\mathbf{m}}(t)$, of opposite Grassman parity, and subject to

$$[\gamma^{\alpha,\mathbf{m}}(s), \beta_{\beta,\mathbf{n}}(t)] = \delta^\alpha_\beta \delta^{\mathbf{m}}_{\mathbf{n}} \delta(s-t). \tag{5.7}$$

$\phi_{\alpha,\mathbf{m}}(t)$ is defined for $|\mathbf{m}| \leqslant p$, but $\mathcal{D}_{\alpha,\mathbf{m}}(t)$ (and thus $\beta_{\alpha,\mathbf{m}}(t)$ and $\gamma^{\alpha,\mathbf{m}}(t)$) are only defined for $|\mathbf{m}| \leqslant p-1$, due to the appearence of $\phi_{\alpha,m+\underline{\mu}}(t)$ in (5.6). Moreover, the t derivative makes the causal weight equal one rather than zero. $DGRO(N,\mathfrak{g})$ thus acts on the antifields as

$$\begin{aligned}
[\mathcal{L}_\xi, \beta_{\alpha,\mathbf{n}}(t)] &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|} \varrho^\beta_\alpha(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t))))\beta_{\beta,\mathbf{m}}(t), \\
[\mathcal{J}_X, \beta_{\alpha,\mathbf{n}}(t)] &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|} \varrho^\beta_\alpha(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t))))\beta_{\beta,\mathbf{m}}(t), \\
[L_f, \beta_{\alpha,\mathbf{n}}(t)] &= -f(t)\dot{\beta}_{\alpha,\mathbf{n}}(t) - \dot{f}(t)\beta_{\alpha,\mathbf{n}}(t),
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
[\mathcal{L}_\xi, \gamma^{\alpha,\mathbf{m}}(t)] &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \gamma^{\beta,\mathbf{n}}(t)\varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t)))), \\
[\mathcal{J}_X, \gamma^{\alpha,\mathbf{m}}(t)] &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \gamma^{\beta,\mathbf{n}}(t)\varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t)))), \qquad (5.9)\\
[L_f, \gamma^{\alpha,\mathbf{m}}(t)] &= -f(t)\dot\gamma^{\alpha,\mathbf{m}}(t).
\end{aligned}$$

Equivalently, the contributions to the $DGRO(N)$ generators are

$$\begin{aligned}
\mathcal{L}^{(\mathcal{D})}_\xi &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \int dt\ \varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t)))) :\beta_{\alpha,\mathbf{m}}(t)\gamma^{\beta,\mathbf{n}}(t):, \\
\mathcal{J}^{(\mathcal{D})}_X &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \int dt\ \varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t)))) :\beta_{\alpha,\mathbf{m}}(t)\gamma^{\beta,\mathbf{n}}(t):, \qquad (5.10)\\
L^{(\mathcal{D})}_f &= \sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \int dt\ f(t) :\beta_{\alpha,\mathbf{m}}(t)\dot\gamma^{\alpha,\mathbf{m}}(t):,
\end{aligned}$$

apart from cohomologically trivial terms. The contribution to the KT generator is

$$Q^{(\mathcal{D})}_{KT} = \sum_{|\mathbf{m}|\leqslant p-1} \int dt\ \mathcal{D}_{\alpha,\mathbf{m}}(t)\gamma^{\alpha,\mathbf{m}}(t). \qquad (5.11)$$

Note that no normal ordering is necessary here. The antifield contribution to the abelian charges is readily computed in terms of the functions (4.33). $\mathfrak{b}_{\alpha,\mathbf{m}}(t)$ has causal weight one, is defined for $|\mathbf{m}| \leqslant p-1$, and has opposite Grassmann parity, so its contribution counts negative. Hence the abelian charges are

$$c_j = c^{(q)}_j(N) + c^{(e)}_j(1) + c^{(\phi)}_j(p, N; \varrho, 0) - c^{(\phi)}_j(p-1, N; \varrho, 1). \qquad (5.12)$$

Instead of (5.6), we can consider the alternative constraint

$$\mathcal{D}_{\alpha,\mathbf{m}}(t) = e^{-1}(t)(\dot\phi_{\alpha,\mathbf{m}}(t) - \dot q^\mu(t)\phi_{\alpha,\mathbf{m}+\underline{\mu}}(t)) \approx 0, \qquad (5.13)$$

where $e^{-1}(t)$ is the inverse of the einbein $e(t)$, corresponding to the equally true fact $e^{-1}(t)\partial\phi_\alpha(x,t)/\partial t = 0$. The expression in (5.13) has causal weight zero. This change induces the following modifications in (5.8), (5.9) and (5.10):

$$\begin{aligned}
[L_f, \beta_{\alpha,\mathbf{n}}(t)] &= -f(t)\dot\beta_{\alpha,\mathbf{n}}(t), \\
[L_f, \gamma^{\alpha,\mathbf{m}}(t)] &= -f(t)\dot\gamma^{\alpha,\mathbf{m}}(t) - \dot f(t)\gamma^{\alpha,\mathbf{m}}(t) \qquad (5.14)\\
L^{(\mathcal{D})}_f &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-1} \int dt\ f(t) :\dot\beta_{\alpha,\mathbf{m}}(t)\gamma^{\alpha,\mathbf{m}}(t):.
\end{aligned}$$

Using (4.34), the abelian charges now become

$$c_j = c_j^{(q)}(N) + c_j^{(e)}(1) + \tilde{c}_j^{(\phi)}(p, N-1; \varrho, 0). \tag{5.15}$$

This result has a natural interpretation: when (5.13) is taken into account, only the transverse modes $\phi_{\alpha,\mathbf{m}}(t) - \dot{q}^\mu(t)\phi_{\alpha,\mathbf{m}+\underline{\mu}}(t)$ contribute. They are equal in number to $\phi_{\alpha,\mathbf{m}}(t)$ with $m_0 = 0$, i.e. the dimension is effectively reduced from N to $N-1$.

Thus the cohomologies of the equivalent constraints (5.6) and (5.13) give rise to inequivalent $DGRO(N, \mathfrak{g})$ modules. A deeper understanding of this disturbing fact is lacking. For definiteness, only the second form (5.13) is considered henceforth, so the abelian charges are given by (5.15).

5.3 Euler-Lagrange constraint

Since the EL constraint (3.8) is a local functional, it can be expanded in a Taylor series,

$$\mathcal{E}^\alpha(x) = \sum_{|\mathbf{m}|\geqslant 0} \frac{1}{\mathbf{m}!}\mathcal{E}^\alpha_{,\mathbf{m}}(t)(x-q(t))^{\mathbf{m}}. \tag{5.16}$$

Assume that the EL equations are of order o_α, which typically has the values $o_\alpha = (2,1,0)$ for ϕ_α a (bosonic, fermionic, auxiliary) degree of freedom. Then the Taylor coefficients $\mathcal{E}^\alpha_{,\mathbf{m}}(t)$ is a function of $\phi_{\beta,\mathbf{n}}(t)$ for $|\mathbf{n}| \leqslant |\mathbf{m}| + o_\alpha$, which is well defined on $J^p\mathcal{Q}$ provided that $|\mathbf{m}| \leqslant p - o_\alpha$. Thus, $\mathcal{E}^\alpha_{,\mathbf{m}}(t)$ transforms as a $(p-o_\alpha)$-jet with an upper V index and a lower multi-index, and with causal weight zero. The EL constraint now takes the form

$$\mathcal{E}^\alpha_{,\mathbf{m}}(t) \approx 0, \qquad \forall\, |\mathbf{m}| \leqslant p - o_\alpha. \tag{5.17}$$

The antifield $(p-o_\alpha)$-jet $\phi^{*\alpha}_{,\mathbf{m}}(t)$ is introduced to kill the EL equation (5.17) in cohomology; it can be considered as the Taylor coefficients of $\phi^{*\alpha}(x)$ up to order $p - o_\alpha$. The corresponding momentum $\pi^{*,\mathbf{m}}_\alpha(t)$ is defined by relations

$$[\pi^{*,\mathbf{m}}_\alpha(s), \phi^{*\beta}_{,\mathbf{n}}(t)] = \delta^\beta_\alpha \delta^{\mathbf{m}}_{\mathbf{n}} \delta(s-t). \tag{5.18}$$

The contributions to the $DGRO(N)$ generators are

$$\begin{aligned}
\mathcal{L}^{(\mathcal{E})}_\xi &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-o_\alpha} \int dt\ \varrho^\alpha_\beta(T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t))))\,{:}\phi^{*\beta}_{,\mathbf{m}}(t)\pi^{*,\mathbf{n}}_\alpha(t){:}\,, \\
\mathcal{J}^{(\mathcal{E})}_X &= -\sum_{|\mathbf{m}|\leqslant|\mathbf{n}|\leqslant p-o_\alpha} \int dt\ \varrho^\alpha_\beta(J^{\mathbf{m}}_{\mathbf{n}}(X(q(t))))\,{:}\phi^{*\beta}_{,\mathbf{m}}(t)\pi^{*,\mathbf{n}}_\alpha(t){:}\,,
\end{aligned} \tag{5.19}$$

$$L_f^{(\mathcal{E})} \;=\; -\sum_{|\mathbf{m}|\leqslant p-o_\alpha} \int dt\; f(t)\,{:}\dot\phi^{*\alpha}_{,\mathbf{m}}(t)\pi^{*,\mathbf{m}}_\alpha(t){:}\,,$$

apart from cohomologically trivial terms.

To make the notation completely clear: the representation ϱ is a direct sum, $\varrho = \varrho_{\mathrm{bosonic}} \oplus \varrho_{\mathrm{fermionic}} \oplus \varrho_{\mathrm{auxiliary}}$, so $\mathcal{L}^{(\mathcal{E})}_\xi = \mathcal{L}^{(\mathcal{E})}_{\xi,\mathrm{bosonic}} + \mathcal{L}^{(\mathcal{E})}_{\xi,\mathrm{fermionic}} + \mathcal{L}^{(\mathcal{E})}_{\xi,\mathrm{auxiliary}}$, and the sums in (5.19) runs over $|\mathbf{n}| \leqslant p-2$, $p-1$, and p, respectively.

Since $\phi^{*\alpha}_{,\mathbf{m}}(t)$ carries an upper V index, it transforms in the dual $gl(N)\oplus\mathfrak{g}$ representation $\varrho^\dagger$, just like $\pi^{\alpha,\mathbf{m}}(t)$. The contribution to the KT generator is

$$Q^{(\mathcal{E})}_{KT} = \sum_{|\mathbf{m}|\leqslant p-o_\alpha} \int dt\; \mathcal{E}^\alpha_{,\mathbf{m}}(t)\pi^{*,\mathbf{m}}_\alpha(t), \tag{5.20}$$

which adds $-c^{(\phi)}(p-o_\alpha, N; \varrho^\dagger, 0)$ to the abelian charges.

However, the constraints (5.13) and (5.17) are not independent, because

$$\mathcal{B}^\alpha_{,\mathbf{m}}(t) = e^{-1}(t)(\dot{\mathcal{E}}^\alpha_{,\mathbf{m}}(t) - \dot q^\mu(t)\mathcal{E}^\alpha_{,\mathbf{m}+\underline{\mu}}(t)) \approx 0, \tag{5.21}$$

for every $\mathbf{m}$ such that $|\mathbf{m}| \leqslant p-o_\alpha-1$. New antifields must therefore be introduced to eliminate the unwanted cohomology; call these $\beta^{*\alpha}_{,\mathbf{m}}(t)$ and their momenta $\gamma^{*,\mathbf{m}}_\alpha(t)$. The contribution to the KT generator is

$$Q^{(\mathcal{B})}_{KT} = \sum_{|\mathbf{m}|\leqslant p-o_\alpha-1} \int dt\; \mathcal{B}^\alpha_{,\mathbf{m}}(t)\gamma^{*,\mathbf{m}}_\alpha(t). \tag{5.22}$$

Similarly, there are contributions to $\mathcal{L}_\xi$, $\mathcal{J}_X$ and L_f, analogous to (5.19), but the sum only runs up to $p-o_\alpha-1$. These antifields have opposite Grassmann parity from $\phi^{*\alpha}_{,\mathbf{m}}(t)$ and $\beta_{\alpha,\mathbf{m}}(t)$, and thus the same parity as the original field $\phi_{\alpha,\mathbf{m}}(t)$. As the notation suggests, we can view them as the antifields of $\beta_{\alpha,\mathbf{m}}(t)$. Their addition to the abelian charges is $c^{(\phi)}(p-o_\alpha-1, N; \varrho^\dagger, 0)$. In view of (4.34), the net contribution to the abelian charges from the antifields is thus

$$\begin{aligned} &-c^{(\phi)}(p-o_\alpha, N; \varrho^\dagger, 0) + c^{(\phi)}(p-o_\alpha-1, N; \varrho^\dagger, 0) \\ &= -\tilde c^{(\phi)}(p-o_\alpha, N-1; \varrho^\dagger, 0). \end{aligned} \tag{5.23}$$

5.4 Geodesic constraint

The geodesic and einbein constraints are modified as follows in the passage to jet space. In (3.33), replace $g_{\mu\nu}(q(t))$ and $\Gamma^\nu_{\sigma\tau}(q(t))$ by the zero-jets $g_{\mu\nu}(t)$

and $\Gamma^\nu_{\sigma\tau}(t)$, and in the definition of the latter (3.28), replace $\partial_\rho g_{\mu\nu}(q(t))$ by $g_{\mu\nu,\rho}(t)$. The trajectory antifield thus contributes $-c^{(q)}_j(N)$ to the abelian charges, which cancels the contribution from the observer's trajectory. The einbein antifield has causal weight zero, but since $c^{(e)}_j(1)-c^{(e)}_j(0)=0$, (both terms equal $2\delta_{j,4}$), the net result from the einbein is zero.

5.5 Noether identities

Finally, we must consider the Noether symmetries. In (3.39), we expand $\mathcal{E}^\alpha(y)$ in a Taylor series. By invariance, it is now clear that there must exist some functions $r^{a,\mathbf{n}}_\alpha(x,t)$, such that

$$r^a(x) = \sum_{\mathbf{n}} \int dt\ r^{a,\mathbf{n}}_\alpha(x,t)\mathcal{E}^\alpha_{,\mathbf{n}}(t). \tag{5.24}$$

Hence the corresponding jet, obtained by a Taylor expansion in x, has the form

$$r^a_{,\mathbf{m}}(s) = \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|} \int dt\ r^{a,\mathbf{n}}_{\alpha,\mathbf{m}}(s,t)\mathcal{E}^\alpha_{,\mathbf{n}}(t). \tag{5.25}$$

This formula defines functions $r^{a,\mathbf{n}}_{\alpha,\mathbf{m}}(s,t)$, whose transformation properties is clear from their index structure. The Noether identities now turn into operator identities $r^a_{,\mathbf{m}}(s)\equiv 0$, valid for all $\mathbf{m}$ of sufficiently low order, say $|\mathbf{m}|\leqslant p-o_a$, where o_a is the order of the original Noether identity (3.39). Thus for every physical state $|phys\rangle$,

$$\sum_{|\mathbf{n}|\leqslant|\mathbf{m}|} \int dt\ (-)^\alpha r^{a,\mathbf{n}}_{\alpha,\mathbf{m}}(s,t)\phi^{*\alpha}_{,\mathbf{n}}(t)|phys\rangle \tag{5.26}$$

is also physical. To eliminate this unwanted cohomology, we must introduce additional (bosonic) Noether antifields to make (5.26) exact. Denote these jets $\mathfrak{b}^a_{,\mathbf{m}}(t)$ and their momenta $\mathfrak{c}^{;\mathbf{m}}_a(t)$. To (3.40) and (3.41) correspond

$$\begin{aligned}
&[\mathfrak{c}^{;\mathbf{m}}_a(s), \mathfrak{b}^b_{,\mathbf{n}}(t)] = \delta^b_a\delta^{;\mathbf{m}}_{,\mathbf{n}}\delta(s-t), &(5.27)\\
&Q^{(2)}_{KT} = \sum_{|\mathbf{n}|\leqslant|\mathbf{m}|\leqslant p-o_a} \iint dsdt\ (-)^\alpha r^{a,\mathbf{n}}_{\alpha,\mathbf{m}}(s,t)\phi^{*\alpha}_{,\mathbf{n}}(t)\mathfrak{c}^{;\mathbf{m}}_a(s).
\end{aligned}$$

Now the state in (5.26) can be written as $Q_{KT}\mathfrak{b}^a_{,\mathbf{m}}(t)|phys\rangle$, and thus it no longer contributes to the cohomology.

Now specialize to $DGRO(N, \mathfrak{g})$. To each type of symmetry, we assign bosonic antifields and momenta according to the following table.

symmetry	antifield	af. momentum	antifield jet	af. momentum jet
$diff(N)$	$\mathfrak{b}_\mu(x)$	$\mathfrak{c}^\mu(x)$	$\mathfrak{b}_{\mu,\mathbf{m}}(t)$	$\mathfrak{c}^{\mu,\mathbf{m}}(t)$
$diff(1)$	$\mathfrak{b}(t)$	$\mathfrak{c}(t)$	$\mathfrak{b}(t)$	$\mathfrak{c}(t)$
$map(N, \mathfrak{g})$	$\mathfrak{b}^a(x)$	$\mathfrak{c}_a(x)$	$\mathfrak{b}^a_{,\mathbf{m}}(t)$	$\mathfrak{c}_a^{;\mathbf{m}}(t)$

The Noether identities for $diff(N)$ and $map(N, \mathfrak{g})$ are both of order three, because the dominating terms are $\partial_\nu G^{\mu\nu}(x) \equiv 0$ and $\partial_\mu\partial_\nu F^{\mu\nu}(x) \equiv 0$, respectively. Therefore, $\mathfrak{b}_{\mu,\mathbf{m}}(t)$ and $\mathfrak{b}^a_{,\mathbf{m}}(t)$ are both defined for $|\mathbf{m}| \leqslant p-3$, while the reparametrization antifield is not affected by the passage to jet space.

Define the following fields:

$$
\begin{aligned}
\mathcal{S}_\mu(x) &= \int d^N y\ (-)^\alpha [\mathcal{L}_\mu(x), \phi_\alpha(y)]\phi^{*\alpha}(y), \\
\mathcal{T}_\mu(x) &= \int dt\ [\mathcal{L}_\mu(x), q^\nu(t)] q^*_\nu(t), \\
\mathcal{W}^a(x) &= \int d^N y\ (-)^\alpha [\mathcal{J}^a(x), \phi_\alpha(y)]\phi^{*\alpha}(y).
\end{aligned}
\tag{5.28}
$$

Note that $\mathcal{T}_\mu(x) \propto \delta^N(x - q(t))$. If $\mathcal{S}_{\mu,\mathbf{m}}(t)$, $\mathcal{T}_{\mu,\mathbf{m}}(t)$, and $\mathcal{W}^a_{,\mathbf{m}}(t)$ denote the corresponding jet space trajectories, the KT generator contributions (3.56) become

$$
\begin{aligned}
Q_{KT}^{(\mathrm{diff})} &= \sum_{|\mathbf{m}|\leqslant p-3} \int dt\ (\mathcal{S}_{\mu,\mathbf{m}}(t) + \mathcal{T}_{\mu,\mathbf{m}}(t))\mathfrak{c}^{\mu,\mathbf{m}}(t), \\
Q_{KT}^{(\mathrm{rep})} &= \iint ds dt\ ([L(s), q^\mu(t)] q^*_\mu(t) + [L(s), e(t)] e^*(t))\mathfrak{c}(s), \\
Q_{KT}^{(\mathrm{gauge})} &= \sum_{|\mathbf{m}|\leqslant p-3} \int dt\ \mathcal{W}^a_{,\mathbf{m}}(t)\mathfrak{c}_a^{;\mathbf{m}}(t),
\end{aligned}
\tag{5.29}
$$

However, the Noether antifields are not all independent, because there are further conditions analogous to (5.13).

$$
\begin{aligned}
e^{-1}(t)(\dot{\mathfrak{b}}_{\mu,\mathbf{m}}(t) - \dot{q}^\mu(t)\mathfrak{b}_{\mu,\mathbf{m}+\underline{\mu}}(t)) &\approx 0, \\
e^{-1}(t)(\dot{\mathfrak{b}}^a_{,\mathbf{m}}(t) - \dot{q}^\mu(t)\mathfrak{b}^a_{\mu,\mathbf{m}+\underline{\mu}}(t)) &\approx 0.
\end{aligned}
\tag{5.30}
$$

We must thus introduce further antifields to eliminate these relations in cohomology.

It is now straightforward to write down the Noether antifield contributions to the DGRO algebra generators. Suffice it to say, that they transform in the adjoint representation of $DGRO(N)$, which corresponds to the $gl(N) \oplus \mathfrak{g}$ representation $(1,0;0) \oplus \mathrm{ad}_\mathfrak{g}$ and causal weight one. Here $\mathrm{ad}_\mathfrak{g}$ denotes the $\mathfrak{g}$ adjoint and $0_\mathfrak{g}$ the trivial $\mathfrak{g}$ representation. This is distributed among the various Noether antifields according to the following table.

symmetry	antifield	$gl(N)$ rep	$\mathfrak{g}$ rep	causal weight
$diff(N)$	$\mathfrak{b}_\mu(x)$	$(1,0;0)$	$0_\mathfrak{g}$	0
$diff(1)$	$\mathfrak{b}(t)$	$(0,0;0)$	$0_\mathfrak{g}$	1
$map(N,g)$	$\mathfrak{b}^a(x)$	$(0,0;0)$	$\mathrm{ad}_\mathfrak{g}$	0

Moreover, antifields for the conditions (5.30) must also be considered.

Hence we get for the abelian charges

$$
\begin{aligned}
c_j^{(\mathrm{diff})} &= c_j^{(\phi)}(p-3, N; (1,0;0) \oplus 0_\mathfrak{g}, 0) - c_j^{(\phi)}(p-4, N; (1,0;0) \oplus 0_\mathfrak{g}, 0) \\
&= \tilde{c}_j^{(\phi)}(p-3, N-1; (1,0;0) \oplus 0_\mathfrak{g}, 0), \\
c_j^{(\mathrm{rep})} &= c_j^{(e)}(1) = 2\delta_{j,4}, \\
c_j^{(\mathrm{gauge})} &= c_j^{(\phi)}(p-3, N; (0,0;0) \oplus \mathrm{ad}_\mathfrak{g}, 0) - c_j^{(\phi)}(p-4, N; (0,0;0) \oplus \mathrm{ad}_\mathfrak{g}, 0) \\
&= \tilde{c}_j^{(\phi)}(p-3, N-1; (0,0;0) \oplus \mathrm{ad}_\mathfrak{g}, 0).
\end{aligned}
\tag{5.31}
$$

These three terms can be summed to yield the following total contribution from the Noether symmetries:

$$
c_j^{(\mathrm{Noether})} = 2\delta_{j,4} + \tilde{c}_j^{(\phi)}(p-3, N-1; (1,0;0) \oplus \mathrm{ad}_\mathfrak{g}, 1). \tag{5.32}
$$

5.6 Finiteness conditions

The total KT generator is thus

$$
Q_{KT} = Q_{KT}^{(\mathcal{D})} + Q_{KT}^{(\mathcal{E})} + Q_{KT}^{(\mathcal{B})} + Q_{KT}^{(\mathcal{G})} + Q_{KT}^{(\mathrm{Noether})}. \tag{5.33}
$$

Similarly, the $DGRO(N, \mathfrak{g})$ generators have analogous antifield contributions, in addition to the original expressions (4.28). Summing up the abelian charges, we find

$$
\begin{aligned}
c_j &= \tilde{c}_j^{(\phi)}(p, N-1; \varrho, 0) - \tilde{c}_j^{(\phi)}(p - o_\alpha, N-1; \varrho^\dagger, 0) + \\
&\quad +2\delta_{j,4} + \tilde{c}_j^{(\phi)}(p-3, N-1; (1,0;0) \oplus \mathrm{ad}_\mathfrak{g}, 1).
\end{aligned}
\tag{5.34}
$$

The coefficients u_j in front of the cohomologically trivial terms have not been discussed. However, it is clear from the above discussion and (4.36) that they satisfy an analogous equation, i.e.

$$\begin{aligned} u_j &= u_j(p, N-1; \varrho, 0) - u_j(p - o_\alpha, N-1; \varrho^\dagger, 0) \\ &\quad + u_j(p-3, N-1; (1,0;0) \oplus \mathrm{ad}_{\mathfrak{g}}, 1) = 0, \end{aligned} \tag{5.35}$$

because $u_j(p, N; \varrho, \lambda) \propto \lambda = 0$.

We have thus shown that $DGRO(N, \mathfrak{g})$ acts on the cohomology $H^\bullet_{state}(Q_{KT}, J^p\mathcal{F})$, with Q_{KT} given by (5.33), and that the abelian charges are (5.34). It is worth noting that these modules are manifestly well defined, since the starting point (i.e. the unconstrained modules constructed in the previous section) contain no infinities and normal ordering does not affect the KT generator. The construction works for all finite p, but the limit $p \to \infty$ may not exist. This may be mathematically satisfactory, but is too isolationistic for physics: the limit expresses the objective reality of events a finite distance away from the observer. A necessary condition for this limit to exist is that the total abelian charges (5.34) have a finite limit.

The conditions (5.34) have not been analyzed in great detail, but some observations are immediate. Since the functions (4.33) are polynomials in p, we can write

$$c_j = a_j^0 + a_j^1 p + \ldots + a_j^n p^n, \tag{5.36}$$

where a_j^k and n depend on N and j but not p. Hence the finiteness conditions take the form $a_j^k = 0$, for all k, $1 \leqslant k \leqslant n(N, j)$.

The trajectory and einbein contributions are independent of p, so they can be ignored. To leading order in p,

$$\binom{N+p}{p} \approx \frac{p^N}{N!}, \quad \binom{N+p}{p-1} \approx \frac{p^{N+1}}{(N+1)!}, \quad \binom{N+p}{p-2} \approx \frac{p^{N+2}}{(N+2)!}. \tag{5.37}$$

The functions in (5.34) take the form

$$\tilde{c}_j^{(\phi)}(p, N; \varrho, \lambda) \approx a_j^n(N; \varrho, \lambda) p^n \tag{5.38}$$

where

j	$n(N,j)$	$a_j^n(N;\varrho,\lambda)$
1	$N+1$	$\mathrm{sd}(\varrho)/(N+1)!$
2	$N+2$	$\mathrm{sd}(\varrho)/(N+1)(N+1)!$
3	$N+1$	$(1-2\lambda)\,\mathrm{sd}(\varrho)/(N+1)!$
4	N	$2(1-6\lambda+6\lambda^2)\,\mathrm{sd}(\varrho)/N!$
5	N	$y(\varrho)/N!$
6	N	$z(\varrho)/N!$
7	$N+1$	$z(\varrho)/(N+1)!$

Now consider the different contributions to c_j. From the original fields we get (5.38), and from the antifields $-a_j^n(N;\varrho^\dagger,\lambda)(p-o_\alpha)^n$. Since $\mathrm{sd}(\varrho^\dagger)=\mathrm{sd}(\varrho)$, $y(\varrho^\dagger)=y(\varrho)$, $z(\varrho^\dagger)=-z(\varrho)$, and $(p-o_\alpha)^n\approx p^n$, the fields and antifields cancel to leading order in p. Cancellation is even exact for auxiliary fields, but for ordinary fields, the Cauchy data survive in cohomology and give rise to subleading terms. Hence the abelian charges are completely dominated for large p by the Noether antifields, which transform in the representation $\varrho=(1,0;0)\oplus\mathrm{ad}_{\mathfrak{g}}$. The factors a_j^n are typically proportional to $\mathrm{sd}(\varrho)=\dim\varrho_{\mathrm{bosonic}}-\dim\varrho_{\mathrm{fermionic}}$, which is positive since we have assumed that there are no fermionic Noether symmetries.

This is a significant problem, because it means that the abelian charges always diverge. Some possible cures are:

1. Introduce fermionic Noether symmetries, i.e. supersymmetry. This is covered in our formalism apart from some additional signs appearing e.g. in (3.41) and (3.46).

2. Introduce fermionic fields without dynamics, so that no antifields cancel their leading behaviour.

3. Dismiss the Noether antifields altogether. Then there are non-zero states of the form (5.26), so the connection to the classical theory in section 3 is looser, but the DGRO algebra still acts on the cohomology groups.

4. Only consider finite p. To each classical action then corresponds the family of well-defined lowest-energy modules $H^g_{state}(Q_{KT},J^p\mathcal{F})$, but the limit $p\to\infty$ is ill defined.

None of these options is satisfactory, but if we nevertheless opt for the third possibility, the leading terms cancel between the fields and antifields,

whereas the subleading terms are of the form

$$c_j \propto o_\alpha \,\mathrm{sd}(\varrho) p^{n-1}, \qquad o_\alpha \,\mathrm{sd}(\varrho) \equiv 2 \dim \varrho_{\mathrm{bosonic}} - \dim \varrho_{\mathrm{fermionic}} = 0. \quad (5.39)$$

Thus, the $p \to \infty$ limit can only exist if there are twice as many bosonic non-auxiliary fields than fermionic ones. Lower-order terms imply further restrictions on the field content. Since the number of conditions grows with N, there is probably an upper critical dimension above which no solutions exist. However, one should not take this result too seriously, since the Noether antifields were discarded.

6 Discussion

In this paper a large class of projective lowest-energy representations of the Noether symmetries in physics has been constructed. It can be considered as a novel approach to quantization, applicable to all systems including gravity (at finite p), although the limit $p \to \infty$ is problematic. It should be stressed that this approach is very conservative. No unobserved physics, such as extra dimensions, higher-dimensional objects, Planck-scale discreteness, or supersymmetry (at least not for finite p), needs to be assumed. Rather, I start from classical physics as formulated in section 3, expand all fields in a Taylor series around the observer's present position, replace Poisson brackets by commutators, and represent the resulting Heisenberg algebra on a unique Fock space.

A unique feature is that the Noether symmetries have consistent quantum representations, i.e. well-defined modules of non-split, abelian, Lie algebra extensions of the classical symmetry algebra. To my knowledge, this issue has never been addressed before, which is not surprising since the first interesting projective $diff(N)$ modules were only discovered in 1994 [6]. In standard canonical quantization of gravity, the constraints do not even classically reproduce $diff(N)$, but only the so-called "Dirac algebra" ([10], page 169).

Another point is the central role played by the observer. True, she is important (albeit in different ways) in both quantum mechanics and general relativity, but she does not enter directly into the core (Schrödinger and Einstein) equations. Here, the passage to jet space introduces the observer directly into the core formalism.

The DGRO algebra contains non-split abelian extensions, which can be viewed as quantum anomalies. Although anomalies often are though of as

harmful, I believe they are necessary and quite useful. In particular, it is sometimes claimed that diffeomorphism invariance implies that all correlation functions are trivial, but this is true only if the abelian charges vanish. Note here the analogy with conformal field theory [7], where all interesting statistical systems have non-zero Virasoro central charge. The analogy is very close, since the conformal algebra in two real dimensions is isomorphic to (two copies of) the diffeomorphism algebra in one complex dimension.

Rudakov's theorem [15] states that all proper $diff(N)$ modules are included in tensor densities; more precisely, there is a one-to-one correspondance between $gl(N)$ and $diff(N)$ irreps, with one exception: totally skew tensor fields, i.e. differential forms, contain submodules of closed differential forms. Hence the classical representations constructed in section 3 are highly reducible. However, this theorem does not apply to lowest-energy modules, since the abelian charges are non-zero.

Many questions remain to be answered. The most disturbing intrinsic problems are the ambiguity in the definition of the antifields, illustrated by the difference between (5.6) and (5.13), and the problems with the limit $p \to \infty$. Another problem concerns the decomposition into irreps; at least, every module decomposes into its bosonic and fermionic parts. It would also be desirable to make contact with the formalism of standard quantum field theory; of course, one must then restrict to the Poincaré subalgbra of $diff(N)$.

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